

# Bicategories for boundary conditions and for surface defects in 3-d TFT

Jürgen Fuchs <sup>a</sup>, Christoph Schweigert <sup>b</sup>, Alessandro Valentino <sup>b</sup>

<sup>a</sup> *Teoretisk fysik, Karlstads Universitet  
Universitetsgatan 21, S-651 88 Karlstad*

<sup>b</sup> *Fachbereich Mathematik, Universität Hamburg  
Bereich Algebra und Zahlentheorie  
Bundesstraße 55, D-20 146 Hamburg*

## ABSTRACT

We analyze topological boundary conditions and topological surface defects in three-dimensional topological field theories of Reshetikhin-Turaev type based on arbitrary modular tensor categories. Boundary conditions are described by central functors that lift to trivializations in the Witt group of modular tensor categories. The bicategory of boundary conditions can be described through the bicategory of module categories over any such trivialization. A similar description is obtained for topological surface defects. Using string diagrams for bicategories we also establish a precise relation between special symmetric Frobenius algebras and Wilson lines involving special defects. We compare our results with previous work of Kapustin-Saulina and of Kitaev-Kong on boundary conditions and surface defects in abelian Chern-Simons theories and in Turaev-Viro type TFTs, respectively.

# 1 Introduction

An insight gained in recent years in the study of quantum field theories is that interesting effects are captured when allowing for codimension-one defects, i.e. interfaces between regions on which two different theories are living. Depending on the application, it is sensible to impose specific kinds of conditions on such interfaces; for instance, in integrable field theories, integrable defects, as considered e.g. in [DMS, BCZ], are naturally of interest. In two-dimensional rational conformal field theories, the study of totally transmissive defect lines (see e.g. [Wa, PZ, BDDO, FFRS2]) has produced structural information about non-chiral symmetries and Kramers-Wannier-type dualities. It has also become apparent that boundaries and defects are close relatives.

In this paper we concentrate on topological quantum field theories in three dimensions (TFTs), specifically on theories that include (compact) Chern-Simons theories. While for the latter subclass a Lagrangian formulation is available, in the general case considered here we work within the combinatorial approach of Reshetikhin and Turaev [RT] type, which associates a TFT to any semisimple modular tensor category. This includes TFTs of Turaev-Viro type as well.

In the three-dimensional situation the simplest codimension-one structures are surfaces that constitute either a defect surface or a two-dimensional boundary. It is worth stressing that here the term boundary refers, as in [KS1], to a brim at which ‘the three-dimensional world ends’. Such brim boundaries must in particular not be confused with the cut-and-paste boundaries that commonly occur (see e.g. [Tu]) in these theories. Boundaries of the latter kind arise when a three-manifold is cut into more elementary three-manifolds with boundaries; accordingly, their function is to account for locality and to allow for sewing, or cut-and-paste, procedures. Both classes of boundaries are geometric boundaries of three-manifolds, but cut-and-paste boundaries come with additional local (chiral) degrees of freedom and can support vector spaces of conformal blocks. In contrast, brim boundaries need not involve any of those structures. Note that the distinction between two different kinds of boundaries is not specific to three dimensions. In two dimensions, in the discussion of so-called open-closed theories, such a distinction is standard; see e.g. [Mo, Sect. 3], where intervals corresponding to in- and out-going open strings are distinguished from “free boundaries” corresponding to the ends of an open string “moving along a D-brane.”

Among the Reshetikhin-Turaev type theories there are in particular TFTs constructed from lattices, which have e.g. been prominent (see, for instance, [FCGK] for a detailed discussion) in the discussion of universality classes of quantum Hall systems. Thus in this particular case, our results may have applications to topological interfaces with gapped excitations between two quantum Hall fluids. Our discussion applies, however, to arbitrary semisimple modular tensor categories and does not rely on any specific aspects of lattice models.

There is no guarantee that for a given quantum field theory a consistent defect or boundary condition exists at all. In particular there can be theories that make perfect sense in the bulk, but cannot be consistently extended to the boundary. On the other hand, if consistent codimension-one defects, or boundary conditions, do exist, they will typically not be unique. It is then natural to study interfaces between such lower-dimensional regions as well, i.e. interfaces of codimension two. In our case of three-dimensional topological field theories, these are generalized Wilson lines. (In other words, the brim boundaries we consider can contain such Wilson lines. In contrast, this is not possible for a cut-and-paste boundaries. On the other

hand, bulk Wilson lines can end on either kind of boundary – in the case of cut-and-paste boundaries, they end on marked points.) Again, such generalized Wilson lines need not exist, but again, if they do exist, then they need not be unique, so that the game can be repeated one step further.

Hereby we arrive at a four-layered structure: At the top level, we associate a topological field theory of Reshetikhin-Turaev type to each three-dimensional part of a stratified three-dimensional manifold. For two-dimensional parts we deal with physical boundaries or with two-dimensional defects, for which we must choose a boundary condition, respectively, in the same spirit, an additional datum that describes the type, or ‘color’ of the defect. Such a datum has been called a *surface operator* in [KS1]; we prefer the term *surface defect* instead. The third layer of structure consists of one-dimensional structures labeled by generalized Wilson lines that separate boundaries or surface defects. And finally, generalized Wilson lines can fuse and split at point-like defects, which may be interpreted as local field insertions and constitute the forth layer of structure.

The basic questions we are addressing in this paper can thus be posed as follows:

- (1) *Given a three-dimensional region with non-empty boundary for which the TFT of Reshetikhin-Turaev type in the interior is labeled by a modular tensor category  $\mathcal{C}$ , what are the data describing the types of topological boundary conditions on the boundary?*
- (2) *Given two three-dimensional regions separated by a two-dimensional interface, for which the TFTs of Reshetikhin-Turaev type in the two regions are labeled by modular tensor categories  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , respectively, what are the data describing the types of topological surface defects on the interface?*

The key in our analysis of these issues is the following process: a Wilson in line in the three-dimensional bulk can be moved “adiabatically” into the boundary or into a defect surface. This has already been studied in [KS1, Sect. 5.2], and a similar process in two dimensions has been considered in [DKR, Sect. 4.1]. A careful analysis of this process allows us to give a complete answer to both questions, including in particular a criterion for the existence of non-trivial solutions. The analysis yields in particular a model-independent generalization of results that have been obtained in [KS1] for abelian Chern-Simons theories using a Lagrangian description.

Our considerations involve mathematical ingredients that, to the best of our knowledge, have not been applied to Reshetikhin-Turaev type TFTs before. Many of them come from higher category theory, like aspects of fusion categories [ENO1, ENOM] and of braided fusion categories [DrGNO], and specifically the notions [DMNO] of central functors and of the Witt group of non-degenerate fusion categories. This group naturally generalizes the classical Witt group of lattices; it has been originally devised as a tool in the classification of modular tensor categories. Since some familiarity with such concepts is required for appreciating our analysis, we collect the pertinent mathematical background in Section 2.

Our results can be summarized as follows.

- (1a) For a boundary adjacent to a three-dimensional region that is labeled by a modular tensor category  $\mathcal{C}$ , and thus with bulk Wilson lines given by  $\mathcal{C}$  as well, the central information about a topological boundary condition  $a$  is contained in the process of moving Wilson lines to the boundary. It is mathematically described by a central functor  $F_{\rightarrow a}: \mathcal{C} \rightarrow \mathcal{W}_a$ , with  $\mathcal{W}_a$  the fusion category of Wilson lines in the boundary with boundary condition  $a$ .

- (1b) A careful distinction between the three-dimensional physics in the bulk and the two-dimensional physics in the boundary allows one to argue that the functor  $F_{\rightarrow a}$  lifts to a (braided) equivalence  $\tilde{F}_{\rightarrow a}: \mathcal{C} \xrightarrow{\sim} \mathcal{Z}(\mathcal{W}_a)$  between the category  $\mathcal{C}$  of bulk Wilson lines and the Drinfeld center of the category  $\mathcal{W}_a$  of boundary Wilson lines.
- (1c) This equivalence implies that a topological boundary condition exists for a TFT labeled by the modular tensor category  $\mathcal{C}$  if and only if the class of  $\mathcal{C}$  in the Witt group of modular tensor categories is trivial. Put differently, topological boundary conditions exist if and only if the modular tensor category  $\mathcal{C}$  is the Drinfeld center of a fusion category.
- (1d) For fixed  $\mathcal{C}$ , the three-layered structure carried by the boundary conditions and their higher-codimension substructures is a bicategory. It naturally encodes e.g. the fusion of (generalized) Wilson lines.

This bicategory can be constructed from any single boundary condition described by a central functor  $F_{\rightarrow a}: \mathcal{C} \rightarrow \mathcal{W}_a$  as the bicategory of module categories over the fusion category  $\mathcal{W}_a$ . The 1-morphisms of this bicategory – i.e. module functors – describe the possible Wilson lines (one-dimensional defects) on the boundary, including their fusion. The 2-morphisms describe the possible junctions of Wilson lines.

- (2) A similar analysis can be performed for surface defects separating TFTs that are labeled by modular tensor categories  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . There are now two different processes of moving bulk Wilson lines from either  $\mathcal{C}_1$  or  $\mathcal{C}_2$  into the defect surface with a fusion category  $\mathcal{W}_d$  of defect Wilson lines. They yield two central functors, which can be combined into a braided equivalence  $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}} \xrightarrow{\sim} \mathcal{Z}(\mathcal{W}_d)$ . Here  $\mathcal{C}_2^{\text{rev}}$  is the modular category with reversed braiding as compared to  $\mathcal{C}_2$ , and  $\boxtimes$  is the Deligne tensor product. Again this equivalence fully captures a surface defect.

Thus topological surface defects exist if and only if  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are in the same Witt class. Again, once one defect is described by an equivalence  $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}} \xrightarrow{\sim} \mathcal{Z}(\mathcal{W}_d)$  of braided fusion categories, the bicategory of all topological surface defects separating  $\mathcal{C}_1$  and  $\mathcal{C}_2$  is given by the bicategory of  $\mathcal{W}_d$ -modules.

The description of boundary conditions and surface defects in terms of module categories that is achieved in this paper allows for a rigorous treatment of related issues. For instance, we can show that all module functors appearing in our theory admit ambidextrous adjunctions, which brings the technology of string diagrams for bicategories to our disposal. This way we can e.g. provide mathematical foundations for the constructions in [KS2]; in particular we prove:

- (3) To every (special) topological surface defect  $S$  separating a TFT labeled by the modular tensor category  $\mathcal{C}$  from itself, string diagrams provide, for any Wilson line separating  $S$  and the transparent surface defect, an explicit construction of a special symmetric Frobenius algebra in  $\mathcal{C}$ . Different Wilson lines give Morita equivalent algebras; we realize the Morita context explicitly in terms of string diagrams.

Before proceeding to the main body of the text, a few further remarks seem to be in order:

- A TFT of Reshetikhin-Turaev type based on a Drinfeld center of a fusion category  $\mathcal{A}$  is, by the results of [BK, TV], equivalent to a TFT of Turaev-Viro type based on  $\mathcal{A}$ . Topological boundary conditions for TFTs of Reshetikhin-Turaev type thus only exist if the TFT admits a Turaev-Viro type description.

- Not surprisingly, the description of boundary conditions and defects in three-dimensional theories is one step higher in the categorical ladder than for two-dimensional theories, e.g. two-dimensional CFTs, for which boundary conditions and defect lines form categories of modules and of bimodules, respectively.

In fact one expects a relation of boundary conditions for the TFT based on the modular tensor category  $\mathcal{C}$  and  $\mathcal{C}$ -module categories. And indeed, as we will explain in Sections 3 and 4, respectively, the existence of a consistent fusion of bulk and boundary Wilson lines requires such a relation. However, *not* every  $\mathcal{C}$ -module category describes a topological boundary condition. Rather, the structure we present involves more stringent requirements that are fulfilled only by a subclass of  $\mathcal{C}$ -module categories. Analogous comments apply to topological surface defects.

- We describe surface defects and boundary conditions as specific objects of a bicategory, not just as isomorphism classes thereof. This opens up the perspective to obtain a vast extension of the entire Reshetikhin-Turaev construction to manifolds with substructures of arbitrary codimension. Here we will not delve into this issue further, but just mention that a first inspection indeed indicates that one can associate the appropriate vector spaces of conformal blocks to cut-and-paste boundaries of such extended manifolds. Any such construction should respect the known relations between topological field theories of Turaev-Viro and of Reshetikhin-Turaev type and therefore be compatible with the kind of construction that is sketched in [KK].
- We obtain our results separately for boundary conditions and for surface defects. A comparison of the results shows that the two situations are related by a ‘folding’ procedure. We thus find a three-dimensional realization of the ‘folding trick’, which in two-dimensional conformal field theory is often invoked as a heuristic tool.
- We finally comment on surface defects separating  $\mathcal{C}$  from itself. For the Deligne product  $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$  of any modular tensor category  $\mathcal{C}$  there exists canonically a braided equivalence to the center of a fusion category, namely to the center of  $\mathcal{C}$  itself,  $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{C})$ . Thus there exist topological surface defects separating the TFT labeled by  $\mathcal{C}$  from itself. Among them there is in particular the *transparent*, or invisible, surface defect whose presence is equivalent to having no interface at all. It corresponds to  $\mathcal{C}$  seen as a module category over itself. The generalized Wilson lines on the transparent surface defect are just the ordinary Wilson lines.

The rest of this paper is organized as follows. We start by providing some mathematical background information in Section 2; the reader already familiar with the relevant aspects of monoidal categories can safely skip this part. Afterwards we present details of our proposal for boundary conditions (Section 3) and surface defects (Section 4). In section 5 we then use the relation between module categories and Lagrangian algebras to show that, in the specific case of abelian Chern-Simons theories, our analysis gives the same results as the Lagrangian analysis of [KS1]. We conclude in Section 6 with a model-independent study that extends the results of [KS2] about the relation between Frobenius algebras in a modular tensor category  $\mathcal{C}$  and generalized Wilson lines separating the transparent surface defect for  $\mathcal{C}$  from an arbitrary surface defect.

## 2 Mathematical preliminaries

We start by summarizing some pertinent mathematical background. By  $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbf{1}, a_{\mathcal{C}}, l_{\mathcal{C}}, r_{\mathcal{C}})$  we denote a monoidal category with tensor product  $\otimes_{\mathcal{C}}$ , tensor unit  $\mathbf{1}$ , associativity constraint  $a_{\mathcal{C}}$ , and left and right unit constraints  $l_{\mathcal{C}}, r_{\mathcal{C}}$  that obey the pentagon and triangle constraints. In our discussion we will, however, usually suppress the associativity and unit constraints altogether, as is justified by the coherence theorem. We work over a fixed ground field  $\mathbb{k}$  that is algebraically closed and has characteristic zero. For definiteness we take  $\mathbb{k}$  to be the field  $\mathbb{C}$  of complex numbers, which is the case relevant for typical applications. All categories are required to be  $\mathbb{k}$ -linear and abelian.

As we are interested in generalizations of the Reshetikhin-Turaev construction, all categories will be finitely semisimple, i.e. all objects are projective, the number of isomorphism classes of simple objects is finite, and the tensor unit is simple. If such a category is also rigid monoidal and has finite-dimensional morphism spaces, it is called a *fusion category*. With some further structure, such categories encode Moore-Seiberg data of chiral conformal field theories. (Examples can be constructed from even lattices, see Section 5.) We are particularly interested in *braided* categories, i.e. monoidal categories  $\mathcal{C}$  endowed with a natural isomorphism from  $\mathcal{C}$  to  $\mathcal{C}$  with the opposite tensor product (i.e. with a commutativity constraint) satisfying the hexagon axioms.

Objects  $U, V$  of a braided fusion category are said to *centralize* each other iff the monodromy  $c_{U,V} \circ c_{V,U}$  is the identity morphism. For  $\mathcal{D}$  a fusion subcategory of a braided fusion category  $\mathcal{C}$ , the *centralizer*  $\mathcal{D}'$  of  $\mathcal{D}$  is the full subcategory of objects of  $\mathcal{C}$  that centralize every object of  $\mathcal{D}$ . A braided fusion category is called *non-degenerate* iff  $\mathcal{C}' \simeq \text{Vect}_{\mathbb{k}}$  [DMNO, Def. 2.1]; a braided fusion category is called *premodular* iff it is equipped with a twist (or, equivalently, with a spherical structure). A premodular category is *modular*, i.e. its braiding is maximally non-symmetric, iff it is non-degenerate [DrGNO, Prop. 3.7].

### 2.1 Module categories

Categorification of the standard notion of module over a ring yields the notion of a module category over a monoidal category. Similarly, the notion of a bimodule category is the categorification of the notion of a bimodule.

#### Definition 2.1.

(i) A (left) *module category* over a monoidal category  $(\mathcal{A}, \otimes_{\mathcal{A}}, \mathbf{1}, a_{\mathcal{A}}, l_{\mathcal{A}}, r_{\mathcal{A}})$  or, in short, an  $\mathcal{A}$ -*module*, is a quadruple  $(\mathcal{M}, \otimes, a, l)$ , where  $\mathcal{M}$  is a  $\mathbb{k}$ -linear abelian category and  $\otimes: \mathcal{A} \times \mathcal{M} \rightarrow \mathcal{M}$  is an exact bifunctor, while  $a = (a_{U,V,M})_{U,V \in \mathcal{A}, M \in \mathcal{M}}$  and  $l = (l_M)_{M \in \mathcal{M}}$  are natural families of isomorphisms  $a_{U,V,M}: (U \otimes_{\mathcal{C}} V) \otimes M \rightarrow U \otimes (V \otimes M)$  and  $l_M: \mathbf{1} \otimes M \rightarrow M$  that satisfy pentagon and triangle axioms analogous to those valid for a monoidal category.<sup>1</sup>

(ii) In the same spirit, for  $(\mathcal{A}_1, \otimes_{\mathcal{A}_1}, \mathbf{1}_1, a_{\mathcal{A}_1}, l_{\mathcal{A}_1}, r_{\mathcal{A}_1})$  and  $(\mathcal{A}_2, \otimes_{\mathcal{A}_2}, \mathbf{1}_2, a_{\mathcal{A}_2}, l_{\mathcal{A}_2}, r_{\mathcal{A}_2})$  monoidal categories, a  $\mathcal{A}_1$ - $\mathcal{A}_2$ -*bimodule category*, or  $\mathcal{A}_1$ - $\mathcal{A}_2$ -*bimodule*, is a tuple  $(\mathcal{X}, \otimes_1, a_1, l_1, \otimes_2, a_2, r_2, b)$ , where  $\mathcal{X}$  is a  $\mathbb{k}$ -linear abelian category,

$$\otimes_1: \mathcal{A}_1 \times \mathcal{X} \rightarrow \mathcal{X} \quad \text{and} \quad \otimes_2: \mathcal{X} \times \mathcal{A}_2 \rightarrow \mathcal{X} \quad (2.1)$$

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<sup>1</sup> For a complete statement of the axioms see e.g. [Os1, Sect. 2.3].



are bifunctors, while  $a_1 = (a_{1;U,V,X})_{U,V \in \mathcal{A}_1, X \in \mathcal{X}}$ ,  $l_1 = (l_{1;X})_{X \in \mathcal{X}}$  and  $a_2 = (a_{2;X,U,V})_{U,V \in \mathcal{A}_2, X \in \mathcal{X}}$ ,  $l_2 = (l_{2;X})_{X \in \mathcal{X}}$  as well as  $(b_{U;X,V})_{U \in \mathcal{A}_1, V \in \mathcal{A}_2, X \in \mathcal{X}}$  are natural families of isomorphisms  $a_{1;U,V,X} : (U \otimes_{\mathcal{A}_1} V) \otimes_1 X \rightarrow U \otimes_1 (V \otimes_1 X)$ ,  $l_{1;X} : \mathbf{1}_1 \otimes_1 X \rightarrow X$ ,  $a_{2;X,U,V} : X \otimes_2 (U \otimes_{\mathcal{A}_2} V) \rightarrow (X \otimes_2 U) \otimes_2 V$ ,  $l_{2;X} : X \otimes_2 \mathbf{1}_2 \rightarrow X$  and  $b_{U;X,V} : (U \otimes_1 X) \otimes_2 V \rightarrow U \otimes_1 (X \otimes_2 V)$  that satisfy pentagon and triangle axioms similar to those valid for a monoidal category.<sup>2</sup>

**Remark 2.2.**

- (i) Very much like a ring is a left module over itself, any monoidal category  $\mathcal{A}$  is naturally a module category over itself; we denote this ‘regular’  $\mathcal{A}$ -module by  $\mathcal{A}_{\mathcal{A}}$ . Also, via  $F \boxtimes A := F(A)$  every category  $\mathcal{A}$  is a module over the monoidal category  $\mathcal{E}nd(\mathcal{A})$  of endofunctors of  $\mathcal{A}$ .
- (ii) Module categories over  $\mathcal{A}$  can be described in terms of algebras in  $\mathcal{A}$ , i.e. objects  $A$  of  $\mathcal{A}$  together with a multiplication morphism  $m : A \otimes A \rightarrow A$  and a unit morphism  $\eta : \mathbf{1} \rightarrow A$  that obey associativity and unit axioms. As usual one introduces a category  $\text{mod-}A$  of right  $A$ -modules in  $\mathcal{A}$ . One easily verifies that the functor  $(U, M) \mapsto U \otimes M$  endows the category  $\text{mod-}A$  with the structure of a module category over  $\mathcal{A}$  [Os1, Sect.3.1]. Conversely, given a module category, algebras can be constructed in terms of internal Homs.
- (iii) Algebras that are not isomorphic can yield equivalent module categories. In fact, there is a Morita theory generalizing the classical Morita theory of algebras over commutative rings.
- (iv) An  $\mathcal{A}$ -module  $\mathcal{M}$  is the same as a monoidal functor from  $\mathcal{A}$  to the monoidal category  $\mathcal{E}nd(\mathcal{M})$  of endofunctors of  $\mathcal{M}$  [Os1, Prop.2.2].
- (v) We recall that for our purposes we assume all categories to be abelian categories enriched over the category of finite-dimensional complex vector spaces and to be finitely semisimple.

Along with module categories there come corresponding notions of functors and natural transformations.

**Definition 2.3.**

- (i) A (strong) *module functor* between two  $\mathcal{A}$ -modules  $\mathcal{M}$  and  $\mathcal{M}'$  is an additive functor  $F : \mathcal{M} \rightarrow \mathcal{M}'$  together with a natural family  $b = (b_{U,M})_{U \in \mathcal{A}, M \in \mathcal{M}}$  of isomorphisms  $b_{U,M} : F(U \otimes M) \rightarrow U \otimes F(M)$  that satisfy pentagon and triangle axioms analogous to those valid for a monoidal functor.
- (ii) A *natural transformation* between two module functors is a natural transformation of  $\mathbb{k}$ -linear additive functors.
- (iii) The corresponding notions for bimodule categories are defined analogously.

There is also an obvious operation of *direct sum* of  $\mathcal{A}$ -modules:  $\mathcal{M} \oplus \mathcal{M}'$  is the Cartesian product of the categories  $\mathcal{M}$  and  $\mathcal{M}'$  with coordinate-wise additive and module structure. An *indecomposable*  $\mathcal{A}$ -module is one that is not equivalent (as  $\mathcal{A}$ -modules, i.e. via a module functor) to a direct sum of two nontrivial  $\mathcal{A}$ -modules. Any  $\mathcal{A}$ -module can be written as a direct sum of indecomposable ones, uniquely up to equivalence.

## 2.2 Bicategories and Deligne products

Given a monoidal category  $\mathcal{A}$ , the collection of all  $\mathcal{A}$ -modules has a three-layered structure, consisting of  $\mathcal{A}$ -modules, module functors, and module natural transformations. This structure

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<sup>2</sup> A complete statement of the axioms can e.g. be found in [Gr, Def.2.10 & Prop.2.12].

cannot be described any longer in terms of a category; we rather need the notion of a *bicategory*, which is pervasive in this paper. A bicategory has three layers of structure: objects, 1-morphisms and 2-morphisms. The composition of 1-morphisms is not necessarily strictly associative, but only up to 2-isomorphisms; if it *is* strictly associative, one calls the bicategory *strict* (or a 2-category). For 2-morphisms there are two different concatenations, referred to as vertical and horizontal compositions. For details about bicategories see e.g. [Ben].

A standard example for a strict bicategory is the one for which objects are small categories, 1-morphisms are functors and 2-morphisms are natural transformations. An example for a non-strict bicategory is the one whose objects are associative algebras, 1-morphisms are bimodules and 2-morphisms are bimodule maps. Here we are interested, for a given monoidal category  $\mathcal{A}$ , in its bicategory  $\mathcal{A}\text{-Mod}$  of modules, having  $\mathcal{A}$ -modules as objects, module functors as 1-morphisms and natural transformations between module functors as 2-morphisms. Similarly, for any pair  $(\mathcal{A}_1, \mathcal{A}_2)$  of monoidal categories there is the bicategory  $\mathcal{A}_1\text{-}\mathcal{A}_2\text{-Bimod}$ .

The universal property of the tensor product of vector spaces allows one to describe bilinear maps in terms of linear maps out of the tensor product. Similarly, the *Deligne tensor product*  $\mathcal{C}_1 \boxtimes \mathcal{C}_2$  [De, Sect. 5] of abelian categories provides a bijection between bifunctors  $F: \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{D}$  and functors  $\hat{F}: \mathcal{C}_1 \boxtimes \mathcal{C}_2 \rightarrow \mathcal{D}$ . If  $\mathcal{C}_1 = A_1\text{-mod}$  is the category of (left, say) modules over a finite-dimensional  $\mathbb{k}$ -algebra  $A_1$  and  $\mathcal{C}_2 = A_2\text{-mod}$ , then  $\mathcal{C}_1 \boxtimes \mathcal{C}_2$  is equivalent to the category of modules over the  $\mathbb{k}$ -algebra  $A_1 \otimes_{\mathbb{k}} A_2$  [De, Prop. 5.5], and if  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are semisimple with simple objects given by  $S_i$  and  $T_j$ , respectively, then  $\mathcal{C}_1 \boxtimes \mathcal{C}_2$  is semisimple as well, with simple objects given by  $S_i \boxtimes T_j$ .

A significant feature of bimodules over a ring is that they admit a tensor product. The Deligne product can be used in a similar way. Given, say, rings  $R_1, R_2$  and  $R_3$ , the tensor product provides us with functors

$$\otimes_{R_2}: R_1\text{-}R_2\text{-bimod} \times R_2\text{-}R_3\text{-bimod} \rightarrow R_1\text{-}R_3\text{-bimod} \quad (2.2)$$

describing ‘mixed’ tensor products. The Deligne tensor product categorifies this feature as well and provides bifunctors between bimodule categories. For details we refer to [EGNO, Sect. 1.46]. For a *commutative* ring  $R$ , the tensor product of two  $R$ -modules is again an  $R$ -module. Braided tensor categories are categorifications of commutative rings. Indeed, if  $\mathcal{C}$  is a *braided* abelian monoidal category, then the Deligne tensor product endows the bicategory  $\mathcal{C}\text{-Mod}$  with a monoidal structure.

Next we notice that for any  $\mathbb{k}$ -algebra  $A$  the space  $\text{End}_A(A_A)$  of module endomorphisms of  $A$  as a module over itself is isomorphic to  $\text{Hom}_{\mathbb{k}}(\mathbb{k}, A)$  and thus to  $A$ . This suggests to study the properties of the category  $\mathcal{E}nd_{\mathcal{A}}(\mathcal{A}_A)$  of module endofunctors of  $\mathcal{A}$  as a module category over itself. Since endofunctors can be composed,  $\mathcal{E}nd_{\mathcal{A}}(\mathcal{A}_A)$  is a monoidal category. Moreover, we have the following categorified version of the classical isomorphism  $\text{End}_A(A_A) \cong A$  of algebras:

**Proposition 2.4.**

*Let  $\mathcal{A}$  be a  $\mathbb{k}$ -linear monoidal category. For any object  $U \in \mathcal{A}$  denote by  $F_U: \mathcal{A}_A \rightarrow \mathcal{A}_A$  the module endofunctor that acts on objects by tensoring with  $U$  from the right,  $F_U(V) := U \otimes V$ . Then the functor*

$$\begin{aligned} F_{\mathcal{A}}: \quad \mathcal{A} &\longrightarrow \mathcal{E}nd_{\mathcal{C}}(\mathcal{A}_A) \\ U &\longmapsto F_U \end{aligned} \quad (2.3)$$

*is an equivalence of monoidal categories.*



*Proof.* We first show that the functor

$$\begin{aligned} G_{\mathcal{A}} : \quad \mathcal{E}nd_{\mathcal{A}}(\mathcal{A}_{\mathcal{A}}) &\longrightarrow \mathcal{A} \\ F &\longmapsto F(\mathbf{1}) \end{aligned} \quad (2.4)$$

is an essential inverse of  $G_{\mathcal{A}}$ . Indeed we have the chain of equalities  $G_{\mathcal{A}} \circ F_{\mathcal{A}}(U) = G_{\mathcal{A}}(F_U) = F_U(\mathbf{1}) = U \otimes \mathbf{1} = U$ , so that  $G_{\mathcal{A}} \circ F_{\mathcal{A}} = Id_{\mathcal{A}}$ . Conversely, for any  $\varphi \in \mathcal{E}nd_{\mathcal{A}}(\mathcal{A}_{\mathcal{A}})$  the functor  $F_{\mathcal{A}} \circ G_{\mathcal{A}}(\varphi) \in \mathcal{E}nd_{\mathcal{A}}(\mathcal{A}_{\mathcal{A}})$  acts on  $U \in \mathcal{A}_{\mathcal{A}}$  as  $(F_{\mathcal{A}} \circ G_{\mathcal{A}}(\varphi))(U) = F_{G_{\mathcal{A}}(\varphi)}(U) = \varphi(\mathbf{1}) \otimes U$ . The unit constraint of the module functor  $\varphi$  then provides a natural isomorphism to the identity functor. It remains to obtain tensoriality constraints for the functor  $F_{\mathcal{A}}$ . The equalities

$$\begin{aligned} F_{\mathcal{A}}(U \otimes V)(W) &= (U \otimes V) \otimes W \\ &\xrightarrow{\alpha} U \otimes (V \otimes W) = F_{\mathcal{A}}(U)(F_{\mathcal{A}}(V)W) = (F_{\mathcal{A}}(U) \circ F_{\mathcal{A}}(V))(W) \end{aligned} \quad (2.5)$$

show that these are afforded by the associativity constraint  $a_{\mathcal{A}}$  of  $\mathcal{A}$ .  $\square$

## 2.3 Drinfeld center and enveloping category

For algebras over fields, a very useful invariant of the Morita class of an algebra is the center. In our situation, i.e. for algebras in a monoidal category  $\mathcal{A}$ , a similar invariant is at hand which still is an algebra, albeit in a category different from  $\mathcal{A}$ , namely in the *Drinfeld center*  $\mathcal{Z}(\mathcal{A})$ . We recall the definition of the Drinfeld center: for  $\mathcal{A}$  a monoidal category, the objects of the category  $\mathcal{Z}(\mathcal{A})$  are pairs  $(U, e_U)$ , where  $U \in \mathcal{C}$  and  $e_U$  is a ‘half-braiding’, i.e. a functorial isomorphism  $e_U : U \otimes - \xrightarrow{\sim} - \otimes U$  satisfying appropriate axioms, see e.g. [Ka, Ch. XIII.4].  $\mathcal{Z}(\mathcal{A})$  has a natural structure of a braided monoidal category. The forgetful functor

$$\begin{aligned} \varphi_{\mathcal{A}} : \quad \mathcal{Z}(\mathcal{A}) &\rightarrow \mathcal{A} \\ (U, e_U) &\mapsto U \end{aligned} \quad (2.6)$$

is a tensor functor.

The *reverse* category of a braided monoidal category  $\mathcal{C}$ , denoted by  $\mathcal{C}^{\text{rev}}$ , is the same category with opposite braiding; if  $\mathcal{C}$  is even a ribbon category, as in all our applications, we also endow it with the opposite twist. The Deligne product

$$\mathcal{C}^e := \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \quad (2.7)$$

is a categorified version of the enveloping algebra  $A^e = A \otimes_{\mathbf{k}} A^{\text{op}}$  of an associative algebra. Accordingly we call  $\mathcal{C}^e$  the *enveloping category* of  $\mathcal{C}$ . And in the same way as the category of  $A$ -bimodules can be described, as an abelian category, in terms of  $A^e$ -modules, the bicategory  $\mathcal{C}_1\text{-}\mathcal{C}_2\text{-Bimod}$  is equivalent to the bicategory  $(\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}})\text{-Mod}$ .

Suppose now that the monoidal category  $\mathcal{C}$  is already braided itself, with braiding  $c$ . Then the braiding provides a functor, actually a braided tensor functor, from  $\mathcal{C}$  into its center  $\mathcal{Z}(\mathcal{C})$  by  $U \mapsto (U, c_{U,-})$ . We also have a braided tensor functor  $\mathcal{C}^{\text{rev}} \rightarrow \mathcal{Z}(\mathcal{C})$ , which is obtained by the opposite braiding:  $U \mapsto (U, c_{-,U}^{-1})$ . Using the universal property of the Deligne tensor product we combine the two functors into a tensor functor

$$\begin{aligned} G_{\mathcal{C}} : \quad \mathcal{C}^e &\longrightarrow \mathcal{Z}(\mathcal{C}) \\ U \boxtimes V &\longmapsto (U \otimes V, e_{U \otimes V}) \end{aligned} \quad (2.8)$$

where

$$e_{U \otimes V}(W) : U \otimes V \otimes W \xrightarrow{id_U \otimes c_{W,V}^{-1}} U \otimes W \otimes V \xrightarrow{c_{U,W} \otimes id_V} W \otimes U \otimes V. \quad (2.9)$$

The functor  $G_{\mathcal{C}}$  has a natural structure of a braided tensor functor. A braided monoidal category is called *factorizable* iff  $G_{\mathcal{C}}$  is an equivalence of braided monoidal categories. Representation categories of finite-dimensional factorizable Hopf algebras in the sense of [Dr] are factorizable.

It is natural to ask under what condition the functor  $G_{\mathcal{C}}$  is a braided equivalence. This is answered by the

**Lemma 2.5.** [Mü2, ENO1]

*For  $\mathcal{C}$  a semisimple ribbon category, the functor  $G_{\mathcal{C}}$  (2.8) is an equivalence between the center  $\mathcal{Z}(\mathcal{C})$  and the enveloping category  $\mathcal{C}^e$  if and only if  $\mathcal{C}$  is a modular tensor category.*

Thus in particular in the context of the Reshetikhin-Turaev construction, which takes as an input a *modular* tensor category, the center and enveloping category of  $\mathcal{C}$  are equivalent as braided categories, including their spherical structure.

For a braided category  $\mathcal{C}$  the obvious functor  $\mathcal{C}^e \rightarrow \mathcal{C}$  factors through the center of  $\mathcal{C}$ : composing the functor  $G_{\mathcal{C}}$  (2.8) with the forgetful functor, we obtain

$$\mathcal{C}^e \rightarrow \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}. \quad (2.10)$$

Hereby  $\mathcal{C}$  becomes a  $\mathcal{C}^e$ -module, and any  $\mathcal{C}$ -module is turned into a  $\mathcal{C}$ -bimodule.

The following assertion shows again that it is appropriate to regard the Drinfeld center as a categorification of the center of an algebra:

**Proposition 2.6.** [ENO2, Thm. 3.1], [Mü1, Rem. 3.18]

*Let  $\mathcal{A}$  and  $\mathcal{B}$  be fusion categories. Their centers  $\mathcal{Z}(\mathcal{A})$  and  $\mathcal{Z}(\mathcal{B})$  are braided equivalent iff their bicategories  $\mathcal{A}\text{-Mod}$  and  $\mathcal{B}\text{-Mod}$  of module categories are equivalent.*

There is a close relation between module categories and the Drinfeld center [ENOM, Sect. 5.1]. For any indecomposable  $\mathcal{A}$ -module  $\mathcal{M}$  over a fusion category  $\mathcal{A}$ , the category  $\text{End}_{\mathcal{A}}(\mathcal{M})$  of  $\mathcal{A}$ -module endofunctors of  $\mathcal{M}$  is a fusion category, and  $\mathcal{M}$  can be regarded as a right  $\text{End}_{\mathcal{A}}(\mathcal{M})$ -module, and thus as an  $\mathcal{A} \boxtimes \text{End}_{\mathcal{A}}(\mathcal{M})^{\text{rev}}$ -module. The  $\mathcal{A} \boxtimes \text{End}_{\mathcal{A}}(\mathcal{M})^{\text{rev}}$ -module endofunctors of this module category can be identified [DMNO, Sect. 2.6] with the functors of tensoring with an object of the Drinfeld center  $\mathcal{Z}(\mathcal{A})$  from the left, or, alternatively, with the functors of tensoring with an object of  $\mathcal{Z}(\text{End}_{\mathcal{A}}(\mathcal{M}))$  from the right. Comparing the two descriptions of these functors gives the following result:

**Proposition 2.7.** [Sc]

*For any module  $\mathcal{M}$  over a fusion category  $\mathcal{A}$  there is a canonical equivalence*

$$\mathcal{Z}(\mathcal{A}) \xrightarrow{\simeq} \mathcal{Z}(\text{End}_{\mathcal{A}}(\mathcal{M})) \quad (2.11)$$

*of braided categories.*

## 2.4 Central functors

In this brief subsection, we recall a notion that will enter crucially into our analysis of boundary conditions and defect surfaces.

**Definition 2.8.** [Bez, Sect. 2.1]

A structure of a *central functor* on a monoidal functor  $F: \mathcal{C} \rightarrow \mathcal{A}$  from a braided monoidal category  $\mathcal{C}$  to a monoidal category  $\mathcal{A}$  is a natural family of isomorphisms

$$\sigma_{U,V} : F(U) \otimes V \xrightarrow{\cong} V \otimes F(U) \quad (2.12)$$

for  $U$  in  $\mathcal{C}$  and  $V$  in  $\mathcal{A}$ , satisfying the following compatibility conditions:

(i) For  $X, X' \in \mathcal{C}$  the isomorphism  $\sigma_{X, F(X')}$  coincides with the composition

$$F(X) \otimes F(X') \cong F(X \otimes X') \cong F(X' \otimes X) \cong F(X') \otimes F(X), \quad (2.13)$$

where the first and the third isomorphisms are the tensoriality constraints of  $F$ , while the middle isomorphism comes from the braiding on  $\mathcal{C}$ .

(ii) For  $Y_1, Y_2 \in \mathcal{A}$  and  $X \in \mathcal{C}$  the composition

$$F(X) \otimes Y_1 \otimes Y_2 \xrightarrow{\sigma_{X, Y_1} \otimes Y_2} Y_1 \otimes F(X) \otimes Y_2 \xrightarrow{Y_1 \otimes \sigma_{X, Y_2}} Y_1 \otimes Y_2 \otimes F(X) \quad (2.14)$$

coincides with the isomorphism  $\sigma_{X, Y_1 \otimes Y_2}$ .

(iii) For  $Y \in \mathcal{A}$  and  $X_1, X_2 \in \mathcal{C}$  the composition

$$\begin{aligned} F(X_1 \otimes X_2) \otimes Y &\cong F(X_1) \otimes F(X_2) \otimes Y \xrightarrow{F(X_1) \otimes \sigma_{X_2, Y}} F(X_1) \otimes Y \otimes F(X_2) \\ &\xrightarrow{\sigma_{X_1, Y} \otimes F(X_2)} Y \otimes F(X_1) \otimes F(X_2) \cong Y \otimes F(X_1 \otimes X_2) \end{aligned} \quad (2.15)$$

coincides with  $\sigma_{X_1 \otimes X_2, Y}$ .

The following result relates central functors into  $\mathcal{A}$  to the Drinfeld center  $\mathcal{Z}(\mathcal{A})$ :

**Lemma 2.9.** [DMNO, Def. 2.4]

A structure of central functor on  $F: \mathcal{C} \rightarrow \mathcal{A}$  is equivalent to a lift of  $F$  to a braided tensor functor  $\tilde{F}: \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$ , i.e. the composition  $\varphi_{\mathcal{A}} \circ \tilde{F}$  with the forgetful functor (2.6) equals  $F$ ,

$$\begin{array}{ccc} & \mathcal{Z}(\mathcal{A}) & \\ & \uparrow \tilde{F} & \\ \mathcal{C} & \xrightarrow{F} & \mathcal{A} \end{array} \quad \begin{array}{c} \downarrow \varphi_{\mathcal{A}} \\ \end{array} \quad (2.16)$$

## 2.5 Lagrangian algebras

In general, for an algebra  $A$  in a fusion category  $\mathcal{A}$  there is no notion of a center, at least not as an object of  $\mathcal{A}$ . This is simply because  $\mathcal{A}$  is not required to be braided, so that there is no natural concept of commuting factors in a tensor product. As it turns out, the Drinfeld center  $\mathcal{Z}(\mathcal{A})$ , which *is* braided, is the right recipient for a notion of a center. Keeping in mind that, in classical algebra, Morita equivalent algebras have isomorphic centers, a center should better be associated to a module category over  $\mathcal{A}$  rather than to an algebra in  $\mathcal{A}$ .

**Definition 2.10.** [DMNO, Defs. 3.1 & 4.6]

- (i) An algebra  $A$  in a monoidal category is called *separable* iff the multiplication morphism splits as a morphism of  $A$ -bimodules.
- (ii) An algebra in a monoidal category that is also a coalgebra is called *special* iff it is separable, with the right-inverse of the product given by a multiple of the coproduct, and the composition  $\varepsilon \circ \eta$  of the counit and unit is non-zero.
- (iii) An *étale* algebra in a braided  $\mathbb{k}$ -linear monoidal category  $\mathcal{C}$  is a separable commutative algebra in  $\mathcal{C}$ .
- (iv) An étale algebra  $A \in \mathcal{C}$  is said to be *connected* (or *haploid*) iff  $\dim_{\mathbb{k}} \text{Hom}(\mathbf{1}, A) = 1$ .
- (v) A *Lagrangian algebra* in a non-degenerate braided fusion category  $\mathcal{C}$  is a connected étale algebra  $L$  in  $\mathcal{C}$  for which the category  $\mathcal{C}_L^0$  of local  $L$ -modules in  $\mathcal{C}$  is equivalent to  $\mathcal{Vect}_{\mathbb{k}}$  as an abelian category.

**Remark 2.11.** (i) A local (or dyslectic) module  $(M, \rho)$  over a commutative algebra  $A$  is an  $A$ -module for which the representation morphism  $\rho$  satisfies  $\rho \circ c_{A,M} \circ c_{M,A} = \rho$  [Pa,KO,FFRS1]. The full subcategory of dyslectic modules is a braided monoidal category.

(ii) The defining property  $\mathcal{C}_L^0 \simeq \mathcal{Vect}_{\mathbb{k}}$  of a Lagrangian algebra is equivalent to the equality  $(\text{FPdim}(L))^2 = \text{FPdim}(\mathcal{C})$  of Perron-Frobenius dimensions [DMNO, Cor. 3.32].

**Proposition 2.12.** [DNO, Cor. 3.8]

For  $\mathcal{C}$  a non-degenerate braided fusion category, equivalence classes of indecomposable  $\mathcal{C}$ -modules are in bijection with isomorphism classes of triples  $(A_1, A_2, \Psi)$  with  $A_1$  and  $A_2$  connected étale algebras in  $\mathcal{C}$  and  $\Psi: \mathcal{C}_{A_1}^0 \xrightarrow{\simeq} \mathcal{C}_{A_2}^{0 \text{ rev}}$  a braided equivalence between the category of local  $A_1$ -modules and the reverse of the category of local  $A_2$ -modules

**Remark 2.13.** Étale algebras can be obtained from central functors and, conversely, central functors from induction along étale functors:

- (i) Given a central functor  $F: \mathcal{C} \rightarrow \mathcal{A}$  from a braided fusion category  $\mathcal{C}$  to a fusion category  $\mathcal{A}$ , denote by  $R_F$  its right adjoint functor. The object  $R_F(\mathbf{1}_{\mathcal{A}})$  then has a canonical structure of connected étale algebra in  $\mathcal{C}$  [DMNO, Lemma 3.5].
- (ii) For  $\mathcal{C}$  a braided fusion category and  $A$  a connected étale algebra in  $\mathcal{C}$ , the induction functor  $\text{Ind}_A: \mathcal{C} \rightarrow \mathcal{C}_A$  that acts as  $U \mapsto U \otimes A$  admits a natural structure of a central functor [DMNO, Sect. 3.4].
- (iii) If in addition  $\mathcal{C}$  is non-degenerate and  $A$  is Lagrangian, then the lift  $\widetilde{\text{Ind}}_A: \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C}_A)$  of the induction functor is a braided tensor equivalence [DMNO, Cor. 4.1(i)].

We are now in a position to relate indecomposable module categories over a fusion category  $\mathcal{A}$  and Lagrangian algebras in its center  $\mathcal{Z}(\mathcal{A})$ . Denote by

$$F: \mathcal{Z}(\mathcal{A}) \xrightarrow{\simeq} \mathcal{Z}(\text{End}_{\mathcal{A}}(\mathcal{M})) \xrightarrow{\varphi} \text{End}_{\mathcal{A}}(\mathcal{M}) \quad (2.17)$$

the composition of the equivalence (2.11) with the forgetful functor. This is, trivially, a central functor, and the image  $A_{\mathcal{M}}$  of the tensor unit of the monoidal category  $\text{End}_{\mathcal{A}}(\mathcal{M})$  under the functor  $R_F$  right adjoint to  $F$  is an étale algebra and, as it turns out, even a Lagrangian algebra.

The following proposition shows that these Lagrangian algebras can be seen as invariants of indecomposable tensor categories.

**Proposition 2.14.** [DMNO, Prop. 4.8]

*For any fusion category  $\mathcal{A}$  there is a bijection between the sets of isomorphism classes of Lagrangian algebras in  $\mathcal{Z}(\mathcal{A})$  and equivalence classes of indecomposable  $\mathcal{A}$ -modules.*

The proof of this statement is based on Proposition 2.6.

## 2.6 The Witt group

One step in the long-standing problem of classifying rational conformal field theories is the classification of modular tensor categories. Recently, the following algebraic structure was established in the wider context of non-degenerate braided fusion categories (i.e. without assuming a spherical structure): The quotient of the monoid (with respect to the Deligne product) of non-degenerate braided fusion categories by its submonoid of Drinfeld centers forms a group that contains as a subgroup the group  $\mathfrak{W}_{\text{pt}}$  of the classes of non-degenerate pointed braided fusion categories [DMNO, Sect. 5.3]. The latter coincides with the classical Witt group [Wi] of metric groups, i.e. of finite abelian groups equipped with a non-degenerate quadratic form. This motivates the

**Definition 2.15.** [DMNO, Defs. 5.1 & 5.5]

- (i) Two non-degenerate braided fusion categories  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are called *Witt equivalent* iff there exists a braided equivalence  $\mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)$  with suitable fusion categories  $\mathcal{A}_1$  and  $\mathcal{A}_2$ .
- (ii) The *Witt group*  $\mathfrak{W}$  is the group of Witt equivalence classes of non-degenerate braided fusion categories.

It is not hard to see [DMNO] that Witt equivalence is indeed an equivalence relation, and that  $\mathfrak{W}$  is indeed an abelian group, with multiplication induced by the Deligne product. The neutral element of  $\mathfrak{W}$  is the class of all Drinfeld centers, and the inverse of the class of  $\mathcal{C}$  is the class of its reverse category  $\mathcal{C}^{\text{rev}}$ .

As we will see below, in our considerations the Witt group  $\mathfrak{W}$  will play an important role. But we will also be interested in the categories themselves rather than in their classes in  $\mathfrak{W}$ . Moreover, in our context, the categories whose Witt classes are relevant are even modular. Accordingly we set:

**Definition 2.16.**

- (i) A modular tensor category  $\mathcal{C}$  is called *Witt-trivial* iff its class in the Witt group  $\mathfrak{W}$  is the neutral element of  $\mathfrak{W}$ .
- (ii) A *Witt-trivialization* of a modular tensor category  $\mathcal{C}$  consists of a fusion category  $\mathcal{A}$  and an equivalence

$$\alpha : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{A}) \tag{2.18}$$

as ribbon categories.

## 3 Bicategories for boundary conditions

We are now ready to formulate our proposal for topological boundary conditions for Reshetikhin-Turaev type topological field theories. Since a topological field theory of Turaev-Viro type based

on a fusion category  $\mathcal{A}$  has a natural description as a TFT of Reshetikhin-Turaev type based on the Drinfeld center  $\mathcal{Z}(\mathcal{A})$ , our results cover TFTs of Turaev-Viro type as well.

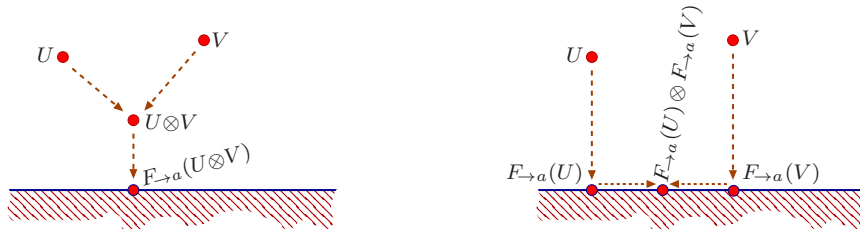
Recall from the Introduction that the boundary conditions we are going to discuss refer to boundaries at which the three-dimensional world ends, rather than cut-and-paste boundaries. As we are working in the Reshetikhin-Turaev framework, in which the categories labeling three-dimensional regions are modular categories and thus in particular finitely semisimple, we only allow for boundary conditions that correspond to finitely semisimple categories as well (though not modular and not even braided, in general, as in two dimensions there is no room for a braiding).

We seize from [KS1, Sect. 5.2] the idea to analyze what happens when Wilson lines in the bulk approach the boundary. We assume that for a given TFT in the bulk there exists a topological boundary condition  $a$  at the end of the three-dimensional world. The two-dimensional boundary can contain Wilson lines. These Wilson lines can carry insertions, and for this reason they are labeled by the objects of a category  $\mathcal{W}_a$ . Boundary Wilson lines can be fused, and accordingly  $\mathcal{W}_a$  has the structure of a monoidal category, and moreover, owing to the fact that the Wilson lines are topological, this comes with dualities. On the other hand, this category is not braided, in general, since there does not exist a natural way to ‘switch’ two boundary Wilson lines without leaving boundary which is two-dimensional.

However, there are Wilson lines in the nearby bulk as well; they are labeled by some modular tensor category  $\mathcal{C}$  (the same that labels the bulk region adjacent to the boundary). The category of bulk Wilson lines is in particular braided, since Wilson lines can be switched in the three-dimensional region. Now part of what is to be meant by a boundary condition is to be able to tell what happens when the boundary is approached from the bulk. Thus we postulate that for a consistent boundary condition there should exist a process of adiabatically moving Wilson lines in the bulk to the boundary, whereby they turn into boundary Wilson lines. Put differently, we postulate that there is a functor

$$F_{\rightarrow a} : \mathcal{C} \rightarrow \mathcal{W}_a. \quad (3.1)$$

Furthermore, the following two processes should yield equivalent results: On the one hand, first fusing two bulk Wilson lines in the bulk and then bringing the so obtained single bulk Wilson line to the boundary; and on the other hand, first moving the two bulk Wilson lines separately to the boundary and then fusing them as boundary Wilson lines inside the boundary. Schematically, showing a two-dimensional section perpendicular to the boundary, the situation looks as follows:



$$(3.2)$$

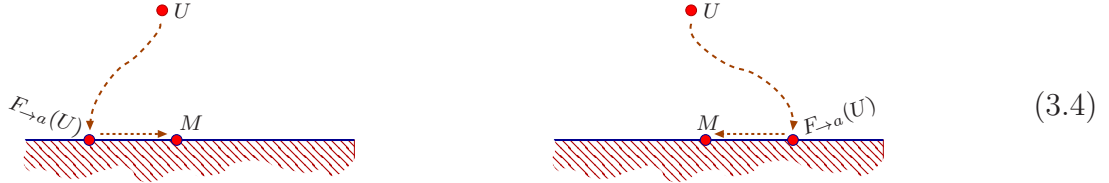
Put differently, the functor (3.1) obeys

$$F_{\rightarrow a}(U \otimes V) \cong F_{\rightarrow a}(U) \otimes F_{\rightarrow a}(V), \quad (3.3)$$



with coherent isomorphisms, i.e. the functor  $F_{\rightarrow a}$  has the structure of a tensor functor. From multiple fusion, one concludes the existence of associativity constraints. Moreover, we should get the same result when homotopies are applied to Wilson lines in the boundary as when they are applied in the bulk. Put differently, the functor  $F_{\rightarrow a}$  should respect dualities.

The next consideration shows that  $F_{\text{bulk} \rightarrow a}$  has even more structure. Consider again the situation that a bulk Wilson line  $U \in \mathcal{C}$  is moved to the boundary, resulting in a boundary Wilson line  $F_{\rightarrow a}(U) \in \mathcal{W}_a$ . Assume in addition that nearby on the boundary there is already another parallel boundary Wilson line  $M \in \mathcal{W}_a$ . Since the process of moving  $U$  to the boundary is supposed to be adiabatic, we should get isomorphic results when we either move  $U$  to the *left* of  $M$  and then fuse  $F(U)$  with  $M$ , or else move  $U$  to the *right* of  $M$  and then fuse  $F(U)$  with  $M$ , as indicated in the following picture.



Put differently, we expect a natural isomorphism

$$F_{\rightarrow a}(U) \otimes M \xrightarrow{\cong} M \otimes F_{\rightarrow a}(U). \quad (3.5)$$

The following argument shows that these isomorphisms endow the functor  $F_{\rightarrow a}$  with the structure of a central functor in the sense of Definition 2.8. Property (i) of a central functor is the statement that for a boundary Wilson line that has been obtained by the adiabatic process, the interchange with another such Wilson line comes from the braiding of bulk Wilson lines. Property (ii) of a central functor, which may be called a boundary Yang-Baxter property, is a consequence of the homotopy equivalence of two different processes in the bulk: either moving the Wilson line in a single step past two boundary Wilson lines, or else doing it in two separate steps. Property (iii) is seen similarly, this time with two bulk Wilson lines involved.

According to lemma 2.9 such a structure is, in turn, equivalent to a lift of  $F_{\text{bulk} \rightarrow a}$  to a braided functor

$$\tilde{F}_{\rightarrow a} : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{W}_a) \quad (3.6)$$

from  $\mathcal{C}$  to the Drinfeld center of the fusion category  $\mathcal{W}_a$ .

The two-dimensional physics of the boundary surface does not provide any natural reason for such a half-braiding rule to exist. The only possible natural origin of such a rule is thus that it is related to the half-braiding in the three-dimensional bulk, via the processes encoded in the functor  $F_{\rightarrow a}$ . Accordingly there should not exist any systematic rule of moving a boundary Wilson line  $M$  to the other side of a neighbouring boundary Wilson line, except through the fact that  $M$  secretly is a bulk Wilson line that has been brought to the boundary (so that the rule comes from the process of first bringing it again into the bulk, moving it around there, and then moving it back to the boundary).

A boundary Wilson line is labeled by an object of  $\mathcal{W}_a$ ; a systematic rule of moving a boundary Wilson line  $M$  to the other side of a neighbouring boundary Wilson line constitutes a half-braiding  $c_{M,-}$  on  $\mathcal{W}_a$  for the object  $M$ . The pair  $(M, c_{M,-})$  is thus just an object in the Drinfeld center  $\mathcal{Z}(\mathcal{W}_a)$ . Put differently, the functor (3.6) is essentially surjective.

Similarly, no information about the bulk should be lost when a bulk Wilson line is brought to the boundary, provided one remembers the way the Wilson line can wander within the bulk to the other side of any other boundary Wilson line. This principle applies likewise to insertions on the Wilson lines. In the bulk, such insertions are morphisms in  $\mathcal{C}$ ; for boundary Wilson lines, we can only allow morphisms that are compatible with the rule to switch the boundary Wilson line with any other boundary Wilson line. In other words, we only allow those morphisms of  $\mathcal{W}_a$  that are compatible with the half-braiding, i.e. we only consider morphisms in  $\mathcal{Z}(\mathcal{W}_a)$ . Put differently, the functor  $\tilde{F}_{\rightarrow a}$  (3.6) is fully faithful, and thus, being also essentially surjective, it is a braided *equivalence*:

$$\tilde{F}_{\rightarrow a} : \mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_a), \quad (3.7)$$

From this equivalence, we conclude that boundary conditions of the type we consider can only consist for a bulk theory that is Witt-trivial in the sense of definition 2.16. Even more, the boundary data are given by a Witt-trivialization.

Once one has understood one boundary condition in a physical theory, frequently the way is open to understand other boundary conditions as well. Thus let us assume that there exists another boundary condition besides  $a$ . At this point we do *not*, however, assume that this boundary condition  $b$  comes with a central functor  $F_{\rightarrow b}$  as well, but rather perform an analysis purely within the boundary. Consider a generalized Wilson line that separates the boundary condition  $a$  on the left from  $b$  on the right. Such Wilson lines can carry local field insertions as well; hence we describe them in terms of a category  $\mathcal{W}_{a,b}$ . We can fuse such a Wilson line with a Wilson line from  $\mathcal{W}_a$  to the left of it. This gives again a Wilson line separating the boundary condition  $a$  from  $b$  and thus an object in  $\mathcal{W}_{a,b}$ . We thus get on the category  $\mathcal{W}_{a,b}$  the structure of a module category over  $\mathcal{W}_a$ .

By a similar argument, the category  $\mathcal{W}_b$  of boundary Wilson lines separating the boundary condition  $b$  from itself has to act on the Wilson lines in  $\mathcal{W}_{a,b}$  from the right. Put differently,  $\mathcal{W}_{a,b}$  is a right  $\mathcal{W}_b$ -module. On the other hand,  $\mathcal{W}_{a,b}$  is already naturally a right module category over the category  $\mathcal{W}_{a,b}^* = \text{End}_{\mathcal{W}_a}(\mathcal{W}_{a,b})$  of module endofunctors. We now invoke a principle of naturality and require that this category describes the tensor category of generalized boundary Wilson lines for the boundary condition  $b$ , i.e. that  $\mathcal{W}_b \simeq \mathcal{W}_{a,b}^*$ .

The latter postulate can only make sense if the fusion category  $\mathcal{W}_{a,b}^*$  comes with a Witt-trivialization of the bulk category  $\mathcal{C}$  as well, i.e. if we have a canonical equivalence

$$\mathcal{C} \simeq \mathcal{Z}(\mathcal{W}_{a,b}^*) \quad (3.8)$$

of braided categories. According to Proposition 2.7 this is indeed the case. This can be seen as a justification of our naturality principle by which we identified Wilson lines with module functors.

To obtain another check of our proposal, we next consider a trivalent vertex in a boundary, with one incoming bulk Wilson line labeled by  $U \in \mathcal{C}$ , one incoming boundary Wilson line labeled by  $W_1 \in \mathcal{W}_a$  and one outgoing boundary Wilson line  $W_2 \in \mathcal{W}_a$ . According to our general picture the three-valent vertex should be labeled by an element of a vector space obtainable as a morphism space. We can realize this vector space in terms of morphisms in the category  $\mathcal{W}_a$ , provided that there is a mixed tensor product

$$\mathcal{C} \times \mathcal{W}_a \longrightarrow \mathcal{W}_a \quad (3.9)$$

and take the trivalent vertex to be labeled by an element of  $\text{Hom}_{\mathcal{W}_a}(U \otimes W_1, W_2)$ . Put differently, we need the structure of a  $\mathcal{C}$ -module on  $\mathcal{W}_a$ . To determine what module category is relevant, we invoke topological invariance of the bulk Wilson line so as to have it running parallel with the boundary before it enters the vertex. We then apply the adiabatic process described by the functor  $F_{\rightarrow a}$  to the piece parallel to the surface, thereby turning the bulk Wilson line with label  $U \in \mathcal{C}$  into a boundary Wilson line with label  $F_{\rightarrow a}(U)$ . This way we reduce the problem of a trivalent vertex involving a bulk Wilson line to the one of a trivalent vertex involving only boundary Wilson lines. The relevant vector space is thus  $\text{Hom}_{\mathcal{W}_a}(F_{\rightarrow a}(U) \otimes W_1, W_2)$ . Put differently, we use the  $\mathcal{C}$ -module structure on  $\mathcal{W}_a$  that is induced by pullback of the regular module category along the monoidal functor  $F_{\rightarrow a} : \mathcal{C} \rightarrow \mathcal{W}_a$  or, what is the same, along the monoidal functor

$$\mathcal{C} \xrightarrow{\tilde{F}_{\rightarrow a}} \mathcal{Z}(\mathcal{W}_a) \xrightarrow{\varphi_{\mathcal{W}_a}} \mathcal{W}_a \quad (3.10)$$

from  $\mathcal{C}$  to  $\mathcal{W}_a$ .

It is important to note that this way one does *not* obtain *all*  $\mathcal{C}$ -modules; thus our results lead to a selection principle that singles out an interesting subclass of  $\mathcal{C}$ -modules. This can be seen already in simple examples, e.g. when  $\mathcal{W}_a$  is the category of finite-dimensional representations of a finite group  $G$ , so that  $\mathcal{Z}(\mathcal{A})$  is the category of finite-dimensional representations of the double  $D(G)$  [Os2, Thm. 3.1]. For instance, for  $G = \mathbb{Z}_2$ , there are two indecomposable  $\mathcal{A}$ -modules (called ‘rough’ and ‘smooth’ in [KK]), but six indecomposable  $\mathcal{Z}(\mathcal{A})$ -modules.

What we have managed so far is to use one given topological boundary condition to obtain also other topological boundary conditions. This raises the question of whether we can obtain *all* topological boundary conditions this way. Suppose we are given two different boundary conditions and thus two braided equivalences

$$\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{A}_1) \quad \text{and} \quad \mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{A}_2). \quad (3.11)$$

Then we have a braided equivalence  $\mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{Z}(\mathcal{A}_2)$ . By the result of [ENO2] on 2-Morita theory of that we recalled in Proposition 2.6, this implies that the bicategories of  $\mathcal{A}_1$ -modules and of  $\mathcal{A}_2$ -modules are equivalent bicategories. We thus conclude that we can indeed access every boundary conditions from any other boundary condition.

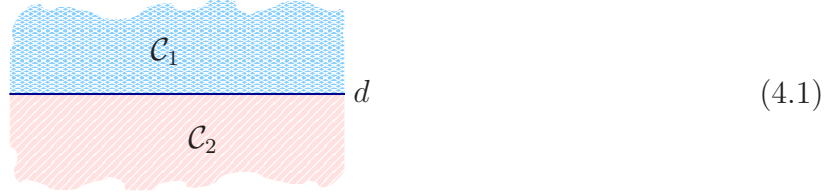
We summarize our proposal: Topological boundary conditions for a topological field theory of Reshetikhin-Turaev type, based on a modular tensor category  $\mathcal{C}$ , are described by Witt-trivializations of  $\mathcal{C}$ , i.e. by braided equivalences  $\mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{A})$ . Given any such trivialization, the bicategory of topological boundary conditions can be identified with the bicategory of  $\mathcal{A}$ -modules.

One should also appreciate that if a TFT of Turaev-Viro type based on the fusion category  $\mathcal{A}$  is described as a Reshetikhin-Turaev theory based on the modular tensor category  $\mathcal{Z}(\mathcal{A})$ , then it comes with a trivialization and the category of topological boundary conditions is naturally identified with the bicategory of  $\mathcal{A}$ -modules. In the special case of TFTs of Turaev-Viro type, our results thus reproduce results of [KK] about boundary conditions in such TFTs.

## 4 Bicategories for surface defects

Next we study what kind of mathematical objects describe topological surface defects, i.e. the topological surface operators considered for abelian Chern-Simons theories in [KS1] or the

domain walls in [BSW, KK]. We consider a surface defect  $d$  separating two modular tensor categories  $\mathcal{C}_1$  and  $\mathcal{C}_2$  and follow the same line of arguments as for boundary conditions in Section 3. The situation to be studied is displayed schematically in the following picture, which shows a two-dimensional section perpendicular to the defect surface:



Again we start with a semisimple fusion category  $\mathcal{W}_d$  of Wilson lines that are contained in the defect surface. We refer to such Wilson lines also as *defect Wilson lines*. In complete analogy with the case of boundary conditions, we postulate that there are adiabatic processes of moving Wilson lines from the bulk on either side of the defect surface into the defect surface, whereby they yield defect Wilson lines. By the same arguments as for boundaries this leads to a central functor

$$F_{\rightarrow d} : \mathcal{C}_1 \rightarrow \mathcal{W}_d \quad (4.2)$$

and, accounting for relative orientations, to another central functor

$$F_{d\leftarrow} : \mathcal{C}_2^{\text{rev}} \rightarrow \mathcal{W}_d. \quad (4.3)$$

According to Lemma 2.9 we thus have two braided functors

$$\tilde{F}_{\rightarrow d} : \mathcal{C}_1 \longrightarrow \mathcal{Z}(\mathcal{W}_d) \quad \text{and} \quad \tilde{F}_{d\leftarrow} : \mathcal{C}_2^{\text{rev}} \longrightarrow \mathcal{Z}(\mathcal{W}_d) \quad (4.4)$$

as in (3.7). Since  $\mathcal{Z}(\mathcal{W}_d)$  is braided, the images of these two functors commute. Thus, with the help of the Deligne tensor product, we combine  $\tilde{F}_{\rightarrow d}$  and  $\tilde{F}_{d\leftarrow}$  into a single functor

$$\tilde{F}_{\rightarrow d\leftarrow} : \mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}} \longrightarrow \mathcal{Z}(\mathcal{W}_d). \quad (4.5)$$

We again invoke a principle of naturality to assert that the combined functor  $\tilde{F}_{\rightarrow d\leftarrow}$  is an equivalence of braided categories.

Suppose now that we have a defect Wilson line  $W \in \mathcal{W}_d$  together with a rule for exchanging  $W$  with any other defect Wilson line  $W' \in \mathcal{W}_d$ . The two-dimensional physics of the defect surface does not provide any natural reason why such a half-braiding rule should exist. The only possible natural origin of such a rule is that it is related to the half-braiding in the three-dimensional parts, using the processes encoded in the two functors  $F_{\rightarrow d}$  and  $F_{d\leftarrow}$ . This amounts to the assumption that the defect Wilson line  $W$  can be written as a direct sum of fusion products of the form  $W_1 \otimes W_2$ , where  $W_1$  is a defect Wilson line that has been obtained by the adiabatic process from  $\mathcal{C}_1$ , i.e.  $W_1 = F_{\rightarrow d}(L_1)$  for some  $L_1 \in \mathcal{C}_1$ , and similarly  $W_2 = F_{d\leftarrow}(L_2)$  with  $L_2 \in \mathcal{C}_2$ . This shows essential surjectivity of  $\tilde{F}_{\rightarrow d\leftarrow}$ ; an argument about point-like insertions on Wilson lines that is completely analogous to one used for boundary conditions shows that  $\tilde{F}_{\rightarrow d\leftarrow}$  is fully faithful.

We thus arrive at an equivalence

$$\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_d) \quad (4.6)$$

of braided categories that, together with the fusion category  $\mathcal{W}_d$ , is part of the data specifying a surface defect. We immediately conclude that a topological surface defect joining regions labeled by the modular tensor categories  $\mathcal{C}_1$  and  $\mathcal{C}_2$  can only exist if  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are in the same Witt class. The existence of such an obstruction should not come as a surprise. Similar effects are, for instance, known from two dimensions: conformal line defects (and, a fortiori, topological defects) can only exist if the two conformal field theories joined by the defect have the same Virasoro central charge. In the situation at hand the Witt group – a concept that has been introduced for independent reasons, namely to structure the space of modular tensor categories – turns out to be the right recipient for the obstruction.

Further, as in the case of boundary conditions, we conclude that other possible labels of surface defects separating  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are described by module categories over the fusion category  $\mathcal{W}_d$ . This also gives the right bicategorical structure to this collection of surface defects: the one of  $\mathcal{W}_d$ -modules. By the same type of argument based on Proposition 2.7 as in the case of boundaries, it follows that the other categories of defect Wilson lines come with Witt-trivializations of  $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}}$  as well, and that the bicategorical structure does not depend on the choice of  $d$ .

Again we can consider two special cases to compare our results with existing literature. The first – abelian Chern-Simon theories – will be relegated to section 5. The second is that the TFT on either side of the defect surface admits a description of Turaev-Viro type, i.e. that both modular tensor categories are Drinfeld centers of fusion categories,  $\mathcal{C}_1 \simeq \mathcal{Z}(\mathcal{A}_1)$  and  $\mathcal{C}_2 \simeq \mathcal{Z}(\mathcal{A}_2)$ . Using the identifications

$$\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}} \simeq \mathcal{Z}(\mathcal{A}_1) \boxtimes \mathcal{Z}(\mathcal{A}_2)^{\text{rev}} \simeq \mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2), \quad (4.7)$$

where we identify left and right half-braidings, shows that in this case the bicategory of  $\mathcal{C}_1$ - $\mathcal{C}_2$  surface defects can be identified with the bicategory of  $\mathcal{A}_1$ - $\mathcal{A}_2$ -bimodules. Thus in the special case of TFTs of Turaev-Viro type our results reproduce those of [KK].

Let us explore some consequences of our results. First we consider the special case that the surface defect separates two regions with the same TFT, i.e. that  $\mathcal{C}_1 = \mathcal{C}_2 =: \mathcal{C}$ . By the characterization of modular tensor categories given in Definition 2.5, there is then a distinguished Witt trivialization,

$$\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C}), \quad (4.8)$$

which is obtained by using the braiding of the categories  $\mathcal{C}$  and of  $\mathcal{C}^{\text{rev}}$ , respectively, to embed them into  $\mathcal{Z}(\mathcal{C})$ . This specific surface defect can be interpreted as a *transparent defect*, very much in the way as a Wilson line labeled by the tensor unit can be seen as a transparent Wilson line (and is, for this reason, usually invisible in a graphical calculus), and accordingly we denote it by the symbol  $T_{\mathcal{C}}$ . Indeed, the defect Wilson lines for this specific defect are labeled by the objects of  $\mathcal{C}$ . The central functor

$$F_{\rightarrow T_{\mathcal{C}}} : \mathcal{C} \rightarrow \mathcal{C} \quad (4.9)$$

describing a specific adiabatic process is, as a functor, just the identity. Its structure of a central functor is then just given by the braiding of  $\mathcal{C}$ . In physical terms this means that in the adiabatic process labels do not change and the braiding is preserved. Similar statements apply to the functor

$$F_{T_{\mathcal{C}} \leftarrow} : \mathcal{C}^{\text{rev}} \rightarrow \mathcal{C}, \quad (4.10)$$

where the structure of a central functor is now given by the opposite braiding. Thus defect Wilson lines separating the surface defect  $T_{\mathcal{C}}$  from itself are naturally identified, including the

braiding, with ordinary Wilson lines in  $\mathcal{C}$ . Phrased the other way round: Wilson lines in the three-dimensional chunk labeled by  $\mathcal{C}$  can be thought of as being secretly Wilson lines inside a defect surface, namely one labeled by the transparent defect  $T_{\mathcal{C}}$ .

We next discuss implications to surface defects separating  $\mathcal{C}$  from itself of the result of [DNO], reported in Proposition 2.12, that indecomposable  $\mathcal{C}$ -modules are in bijection to pairs  $(A_1, A_2)$  of étale algebras in  $\mathcal{C}$  together with a braided equivalence between their full subcategories of local (or dyslectic) modules in  $\mathcal{C}$ . This has the following physical interpretation: A generic Wilson line in  $\mathcal{C}$  cannot pass through a given surface defect. If, however, a whole package of  $\mathcal{C}$ -Wilson lines condenses so as to form a local  $A_1$ -module, the resulting Wilson line can pass through the surface defect and reappear on the other side as a condensed package of  $\mathcal{C}$ -Wilson lines that forms a local  $A_2$ -module.

This is exactly the type of structure needed in the application of surface operators in the TFT construction of the correlators of rational conformal field theories [FRS], following the suggestions of [KS2]. The process then puts the fact [MS, Sect. 4] that the general structure of the bulk partition function is “automorphism on top of extension” in the appropriate and complete setting. This picture can be easily extended to heterotic theories, for which left- and right-moving degrees are in different module tensor categories  $\mathcal{C}_l$  and  $\mathcal{C}_r$ . In particular, the obstruction to the existence of a heterotic TFT construction based on a pair  $(\mathcal{C}_l, \mathcal{C}_r)$  of modular tensor categories is again captured by the Witt group:  $\mathcal{C}_l$  and  $\mathcal{C}_r$  must lie in the same Witt class.

The transmission of (bunches of) Wilson lines should be seen as a three-dimensional analogue of the following process in two dimensions: A topological defect line can wrap around bulk insertions in one full conformal field theory to produce a bulk insertion in another theory. This effect associates a map on bulk fields to any topological defect line. This map has, in turn, been instrumental in obtaining classification results for defects [FGRS] and in understanding their target space formulation [FSW]. We expect that the transmission of Wilson lines can be used to a similar effect in the situation at hand. That the transmission data describe, by Proposition 2.14 the isomorphism class of a module category, is an encouragement for attempting similar classifications as in the two-dimensional case. A first example of a classification will be presented in Section 5.

Returning to the case of general pairs  $(\mathcal{C}_1, \mathcal{C}_2)$  of modular tensor categories, the forgetful functor  $\varphi_{\mathcal{W}_d}$  from the Drinfeld center  $\mathcal{Z}(\mathcal{W}_d)$  to the fusion category  $\mathcal{W}_d$  provides us with a tensor functor

$$\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{W}_d) \xrightarrow{\varphi_{\mathcal{W}_d}} \mathcal{W}_d. \quad (4.11)$$

Via pullback along this functor, the category  $\mathcal{W}_d$  of defect Wilson lines comes with a natural structure of a  $\mathcal{C}_1$ - $\mathcal{C}_2$ -bimodule category. This bimodule structure arises naturally when one considers three-valent vertices in the defect surface with two defect Wilson lines and one bulk Wilson line involved. This structure should also enter in the description of fusion of topological surface defects. We leave a detailed discussion of fusion to future work and only remark that the transparent defect  $T_{\mathcal{C}}$  must act as the identity under fusion.

We conclude with a word of warning: While the structure of a  $\mathcal{C}_1$ - $\mathcal{C}_2$ -bimodule on the category  $\mathcal{W}_d$  of defect Wilson lines can be expected to have a bearing on fusion, the bicategory  $\mathcal{C}_1$ - $\mathcal{C}_2$ -*Bimod* of  $\mathcal{C}_1$ - $\mathcal{C}_2$ -bimodules *cannot* provide the proper mathematical model for the bicategory of surface defects. For instance, taking  $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$ , the natural candidate for the transparent defect is  $\mathcal{C}$  as a bimodule over itself. Using  $\mathcal{C}_1$ - $\mathcal{C}_2$ -*Bimod* as a model for the sur-



face defects, the Wilson lines separating this transparent defect from itself would correspond to bimodule endofunctors of  $\mathcal{C}$ , and the category of these endofunctors is equivalent to  $\mathcal{Z}(\mathcal{C})$  and thus,  $\mathcal{C}$  being modular, to the enveloping category  $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ . We would then *not* recover ordinary Wilson lines as defect Wilson lines in the transparent defect. As we will see in the next section, taking  $\mathcal{C}_1\text{-}\mathcal{C}_2\text{-Bimod}$  as the model for the bicategory of surface defects would also contradict results of [KS1] for abelian Chern-Simons theories, and of [KK] on surface defects in TFTs admitting a description of Turaev-Viro type.

## 5 Lagrangian algebras and abelian Chern-Simons theories

In this section we describe consequences of our proposal in the special case of abelian Chern-Simons theories and compare our findings with the results of [KS1] for this subclass of TFTs. As a new ingredient our discussion involves Lagrangian algebras in the modular tensor category  $\mathcal{C}$  that labels the TFT. Recall from proposition 2.14 that Lagrangian algebras in the Drinfeld center of a fusion category  $\mathcal{A}$  are complete invariants of equivalence classes of indecomposable  $\mathcal{A}$ -module categories.

If we are just interested in equivalence classes of indecomposable boundary conditions, Lagrangian algebras can be used as follows. The presence of a topological boundary condition for a modular tensor category  $\mathcal{C}$  requires the existence of a Witt-trivialization  $\mathcal{C} \simeq \mathcal{Z}(\mathcal{A})$  with  $\mathcal{A}$  a fusion category providing a reference boundary condition. Indecomposable or elementary boundary conditions are then in bijection with indecomposable  $\mathcal{A}$ -modules. The latter, in turn, are in bijection with Lagrangian algebras in the Drinfeld center  $\mathcal{Z}(\mathcal{A})$ , which is just  $\mathcal{C}$ . We can thus classify elementary topological boundary conditions by classifying Lagrangian algebras in  $\mathcal{C}$ . This is of considerable practical interest, since it acquits us of the task to find an explicit Witt-trivialization. However, for many explicit constructions it will be important to have the full bicategorical structure at our disposal, and this requires an explicit Witt-trivialization.

The situation for topological surface defects separating modular tensor categories  $\mathcal{C}_1$  and  $\mathcal{C}_2$  is analogous: the classification of equivalence classes amounts to classifying Lagrangian algebras in  $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}}$ . Again, this avoids finding a Witt-trivialization, but does not give direct access to the full bicategorical structure.

To make contact to the situation studied in [KS1] we first recall some basic facts about abelian Chern-Simons theories and their relation to finite groups with quadratic forms. Let  $\Lambda$  be a free abelian group of rank  $n$  and  $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$  the corresponding real vector space. Denote by  $\mathbb{T}_{\Lambda}$  the torus  $V/\Lambda$ . The classical abelian Chern-Simons theory with structure group  $\mathbb{T}_{\Lambda}$  is completely determined by the choice of a symmetric bilinear form  $K$  on  $V$  whose restriction to the additive subgroup  $\Lambda$  is integer-valued and even. We call the pair  $(\Lambda, K)$  an even *lattice* of rank  $n$ .

### Definition 5.1.

(i) A *bicharacter*, with values in  $\mathbb{C}^{\times}$ , on a finite abelian group  $D$  is a bimultiplicative map  $\beta: D \times D \rightarrow \mathbb{C}^{\times}$ .

A *symmetric* bicharacter, or symmetric bilinear form, on  $D$  is a bicharacter  $\beta$  satisfying  $\beta(x, y) = \beta(y, x)$  for all  $x, y \in D$ .

(ii) A *quadratic form* on a finite abelian group  $D$  is a function  $q: D \rightarrow \mathbb{C}^\times$  such that  $q(x) = q(x^{-1})$  and such that  $\beta(x, y) := q(x \cdot y) / q(x) q(y)$  is a symmetric bilinear form. A *quadratic group* is a finite abelian group endowed with a quadratic form.

To the lattice  $(\Lambda, K)$  we associate a finite group with a quadratic form in the following way. Set  $\Lambda^* := \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R})$ , and denote by  $\text{Im } K$  the image of  $\Lambda$  in  $\Lambda^*$  under the canonical map  $K: \Lambda \rightarrow \Lambda^*$ . The finite abelian group  $D := \Lambda^* / \text{Im } K$  is called the *discriminant group* of the lattice  $(\Lambda, K)$ . Since the symmetric bilinear form  $K$  is integer and even, the group  $D$  comes equipped with a quadratic form  $q: D \rightarrow \mathbb{C}^\times$ , with  $q(\mu) = \exp(2\pi i K(\mu, \mu))$ .

Different lattices may give rise to discriminant groups that are isomorphic as quadratic groups. As argued in [BM], many properties of quantum Chern-Simons theory are encoded in the pair  $(D, q)$ . The pair  $(D, q)$  determines, in turn, an equivalence class of braided monoidal categories, which we denote by  $\mathcal{C}(D, q)$ . For completeness we briefly give some details on how this category is constructed (for more details see [JS, KS1]). First, recall that for any abelian group  $A$  there is a bijection

$$H_{\text{ab}}^3(A; \mathbb{C}^\times) \xrightarrow{\cong} \text{Quad}(A), \quad (5.1)$$

between the group  $\text{Quad}(A)$  of quadratic forms on  $A$  taking values in  $\mathbb{C}^\times$  and the third abelian cohomology group  $H_{\text{ab}}^3(A; \mathbb{C}^\times)$  of  $A$  [Ma, Thm. 3]. A representative for a class in  $H_{\text{ab}}^3(A; \mathbb{C}^\times)$  is given by a pair  $(\psi, \Omega)$  consisting of a 3-cocycle  $\psi$  in the ordinary group cohomology of  $A$  and a 2-cochain  $\Omega$ , satisfying some compatibility conditions (which imply the validity of the hexagon axioms for the braiding (5.2) below). Given a pair  $(\psi, \Omega)$  representing an abelian 3-cocycle, we obtain a quadratic form  $q$  on  $A$  by setting  $q(a) := \Omega(a, a)$  for  $a \in A$ . This realizes one direction of the isomorphism (5.1).

On the other hand, given a quadratic form  $q$  on  $A$ , we obtain a pre-image  $(\psi, \Omega)$  only upon additional choices; one possible choice is an ordered set of generators of the abelian group  $A$ . We will ignore this subtlety in the following and omit the label  $(\psi, \Omega)$  from the notation.

Consider now a quadratic group  $(D, q)$  and choose an abelian 3-cocycle  $(\psi, \Omega)$  representing the quadratic form  $q$  in  $H_{\text{ab}}^3(D; \mathbb{C}^\times)$ . As an abelian category,  $\mathcal{C}(D, q)$  is the category of finite-dimensional complex  $D$ -graded vector spaces and graded linear maps. The simple objects of this category are complex lines  $\mathbb{C}_x$  labeled by group elements  $x \in D$ . In particular we have  $\text{Hom}(\mathbb{C}_x, \mathbb{C}_x) \cong \mathbb{C}$ . We equip the category  $\mathcal{C}(D, q)$  with the tensor product of complex vector spaces, but with associator given by the 3-cocycle  $\psi$ . The 2-cochain  $\Omega$  induces a braiding  $c$  on this monoidal category; the braiding acts on simple objects as

$$\begin{aligned} c_{xy}: \quad \mathbb{C}_x \otimes \mathbb{C}_y &\xrightarrow{\cong} \mathbb{C}_y \otimes \mathbb{C}_x \\ v \otimes w &\longmapsto \Omega(x, y) w \otimes v. \end{aligned} \quad (5.2)$$

The braided pointed fusion category thus obtained depends, up to equivalence of braided monoidal categories, only on the class  $[(\psi, \Omega)]$  in abelian cohomology [JS].

Taking the reverse category amounts to replacing the quadratic form  $q$  by the quadratic form  $q^{-1}$  which takes inverse values, i.e.  $(\mathcal{C}(D, q))^{\text{rev}} \cong \mathcal{C}(D, q^{-1})$ , while the Deligne product amounts to taking the direct sums of the groups and of the quadratic forms. In other words, one has

**Lemma 5.2.** *Let  $(D_1, q_1)$  and  $(D_2, q_2)$  be finite groups with quadratic forms. Then*

$$\mathcal{C}(D_1, q_1) \boxtimes \mathcal{C}(D_2, q_2)^{\text{rev}} \simeq \mathcal{C}(D_1 \oplus D_2, q_1 \oplus q_2^{-1}) \quad (5.3)$$

*as braided monoidal categories.*

A quadratic form  $(D, q)$  is said to be *non-degenerate* iff the associated symmetric bilinear form is non-degenerate in the sense that the associated group homomorphism  $D \rightarrow \text{Hom}(D, \mathbb{C}^\times)$  is an isomorphism. A basic fact about categories of the type  $\mathcal{C}(D, q)$  is

**Lemma 5.3.** [DMNO, Sec. 5.3.]

*The braided monoidal category  $\mathcal{C}(D, q)$  is modular iff the quadratic form  $q$  is non-degenerate.*

In the present context the role of the modular tensor category  $\mathcal{C}(D, q)$  is as the category of (bulk) Wilson lines in the Chern-Simons theory corresponding to a lattice with discriminant group  $(D, q)$ . We now make our proposal for boundary conditions and surface defects explicit in this case. To this end we need an explicit description of Lagrangian algebras.

**Definition 5.4.** [ENOM, Sect. 2.4]

- (i) A *metric group* is a quadratic group  $(D; q)$  for which the quadratic form  $q$  is non-degenerate.
- (ii) For  $U$  a subgroup of a quadratic group  $(D, q)$  with symmetric bilinear form  $\beta: D \times D \rightarrow \mathbb{C}^\times$ , the *orthogonal complement*  $U^\perp$  of  $U$  is the set of all  $d \in D$  such that  $\beta(g, u) = 1$  for all  $u \in U$ . If  $q$  is non-degenerate,  $U^\perp$  is isomorphic to  $D/U$ , so that  $|D| = |U| \cdot |U^\perp|$ .
- (iii) Let  $(D, q)$  be a metric group. A subgroup  $U$  of  $D$  is said to be *isotropic* iff  $q(u) = 1$  for all  $u \in U$ .
- (iv) For any isotropic subgroup  $U$  of a metric group  $(D, q)$  there exists an injection  $U \hookrightarrow (D/U)^*$ , so that  $|U|^2 \leq |D|$ . An isotropic subgroup  $L$  of  $D$  is called *Lagrangian* iff  $|L|^2 = |D|$ .

The concept of a Lagrangian subgroup is linked to Lagrangian algebras by the following assertion, which is a corollary of the results in [DrGNO, Sect. 2.8].

**Theorem 5.5.** *Let  $D$  be a finite abelian group with a nondegenerate quadratic form  $q$ . There is a bijection between Lagrangian subgroups of  $D$  and Lagrangian algebras in  $\mathcal{C}(D, q)$ .*

We thus arrive at the following two statements:

- (1) Elementary topological boundary conditions for the abelian Chern-Simons theory based on the modular tensor category  $\mathcal{C}(D; q)$  are in bijection with Lagrangian algebras in  $\mathcal{C}(D; q)$  and thus with Lagrangian subgroups of the metric group  $(D, q)$ .
- (2) Elementary topological surface defects separating the abelian Chern-Simons theories based on the modular tensor categories  $\mathcal{C}(D_1; q_1)$  and  $\mathcal{C}(D_2; q_2)$  are in bijection with Lagrangian subgroups of the metric group  $(D_1 \oplus D_2, q_1 \oplus q_2^{-1})$ .

The first of these results was established in [KS1] by an explicit analysis using Lagrangian field theory. The second result was then deduced from the first by arguments based on the folding trick.

As a particular case, consider the transparent surface defect  $T_{\mathcal{C}}$  separating  $\mathcal{C}(D, q)$  from itself. It corresponds to the canonical trivialization  $\mathcal{C}(D, q) \boxtimes \mathcal{C}(D, q)^{\text{rev}} \simeq \mathcal{Z}(\mathcal{C}(D, q))$ . The Lagrangian algebra corresponding to  $T_{\mathcal{C}}$  is given by the Cardy algebra  $\bigoplus_{X \in \text{Irr}(\mathcal{C})} X \boxtimes X^\vee$  and corresponds to the diagonal subgroup in  $D \oplus D$ , in accordance with the results in [KS1, Section 3.3].

## 6 Relation with special symmetric Frobenius algebras

In this section we explain how in our framework special symmetric Frobenius algebras can be obtained from certain surface defects  $S$  separating a modular tensor category  $\mathcal{C}$  from itself and a Wilson line separating  $S$  from the transparent defect  $T_{\mathcal{C}}$ . Our results provide a rigorous mathematical foundation for the ideas of [KS2]. A central tool in our study are string diagrams.

### 6.1 String diagrams

A *string diagram* is a planar diagram describing morphisms in a bicategory. Such diagrams are Poincaré dual to another type of diagram frequently used for bicategories, in which 2-morphisms are attached to 2-dimensional parts of the diagram. String diagrams are particularly convenient for encoding properties of adjointness and biadjointness in a graphical calculus. String diagrams apply in particular to the bicategory of small categories in which 1-morphisms are functors and 2-morphisms are natural transformations, an example the reader might wish to keep in mind. For more details see e.g. [La, Kh].

We fix a bicategory; in a first step, we only consider objects and 1-morphisms. They can be visualized in one-dimensional diagrams, with one-dimensional segments describing objects and zero-dimensional parts indicating 1-morphisms. In our convention, such diagrams are drawn horizontally and are to be read from right to left. Thus for  $\mathcal{A}$  and  $\mathcal{B}$  objects of the bicategory and a 1-morphism  $F: \mathcal{A} \rightarrow \mathcal{B}$ , we draw the diagram

$$\text{-----} \mathcal{B} \quad F \quad \mathcal{A} \text{-----} \quad (6.1)$$

The composition  $F_n \cdots F_1 \equiv F_n \circ \cdots \circ F_1: \mathcal{A}_1 \rightarrow \mathcal{A}_n$  of 1-morphisms  $F_i: \mathcal{A}_i \rightarrow \mathcal{A}_{i+1}$  is represented by horizontal concatenation

$$\mathcal{A}_{n+1} \quad F_n \quad \mathcal{A}_n \quad \cdots \quad \mathcal{A}_3 \quad F_2 \quad \mathcal{A}_2 \quad F_1 \quad \mathcal{A}_1 \quad (6.2)$$

To accommodate also 2-morphisms a second dimension is needed. Objects are now represented by two-dimensional regions and 1-morphisms by one-dimensional vertical segments, while zero-dimensional parts indicate 2-morphisms. In our convention, the vertical direction is to be read from bottom to top. Thus a 2-morphism  $\alpha: F_1 \Rightarrow F_2$  between 1-morphisms  $F_1, F_2$  from objects  $\mathcal{A}$  to  $\mathcal{B}$  is depicted by the diagram

$$\begin{array}{ccc} & F_2 & \\ \text{-----} & | & \text{-----} \\ \mathcal{B} & \alpha \bullet & \mathcal{A} \\ & | & \\ \text{-----} & F_1 & \end{array} \quad (6.3)$$

For the moment, we require that the strands always go from bottom to top and do not allow ‘U-turns’. For the identity 2-morphism  $\alpha = id_F$  we omit the blob in the diagram. For the

identity 1-morphism  $Id_{\mathcal{A}}$  we omit any label except for the one referring to the object  $\mathcal{A}$ . With these conventions, 2-morphisms  $\alpha: F \Rightarrow Id_{\mathcal{A}}$  and  $\beta: Id_{\mathcal{A}} \Rightarrow F$  with  $F$  an endo-1-morphism of the object  $\mathcal{A}$  are drawn as

$$\begin{array}{ccc} \text{---} & & \text{---} \\ & \mathcal{A} & \text{and} & \mathcal{A} \\ & \alpha & & \beta \\ & | & & | \\ \text{---} & F & & \text{---} \\ & F & & F \end{array} \quad (6.4)$$

respectively, while a natural transformation  $F_2 F_1 \Rightarrow Id_{\mathcal{A}}$  is represented by

$$\begin{array}{c} \text{---} \\ \mathcal{A} \\ \alpha \\ \text{---} \\ F_2 \quad F_1 \end{array} \quad (6.5)$$

The 2-morphisms of a bicategory can be composed horizontally and vertically. Horizontal composition is depicted as juxtaposition, as in

$$\alpha \otimes \beta = \begin{array}{ccc} \text{---} & & \text{---} \\ & \alpha & \beta \\ & | & | \\ \text{---} & & \text{---} \end{array} \quad (6.6)$$

Vertical composition is represented as vertical concatenation of diagrams; thus e.g.

$$(id_G \otimes \beta) \circ (\alpha \otimes id_F) = \begin{array}{ccc} \text{---} & G & \text{---} \\ & \beta & \\ & | & \\ \text{---} & \alpha & \text{---} \\ & F & \end{array} \quad (6.7)$$

In the bicategory of small categories, we have the notion of an adjoint functor. This notion can be generalized to any 1-morphism in a bicategory. Given two 1-morphisms  $F: \mathcal{A} \rightarrow \mathcal{B}$  and  $G: \mathcal{B} \rightarrow \mathcal{A}$ ,  $G$  is said to be *right adjoint* to  $F$ , and  $F$  *left adjoint* to  $G$ , iff there exist 2-morphisms

$$\eta: Id_{\mathcal{A}} \Rightarrow GF \quad \text{and} \quad \varepsilon: FG \Rightarrow Id_{\mathcal{B}} \quad (6.8)$$

satisfying

$$(id_F \otimes \eta) \circ (\varepsilon \otimes id_F) = id_F \quad \text{and} \quad (\eta \otimes id_G) \circ (id_G \otimes \varepsilon) = id_G. \quad (6.9)$$

The 2-morphisms  $\eta$  and  $\varepsilon$ , if they exist, are not unique. For any number  $\lambda \in \mathbb{C}^\times$  we can replace  $\eta$  by  $\lambda \eta$  and  $\varepsilon$  by  $\lambda^{-1} \varepsilon$  to get another pair of morphisms. For each such pair,  $\eta$  is called a *unit* and  $\varepsilon$  a *counit* of the *adjoint pair*  $(F, G)$ .

In the diagrammatic description, special notation is introduced for the unit and counit of an adjoint pair of functors: we depict them as

$$\eta = \begin{array}{c} \text{---} G \quad F \text{---} \\ \quad \searrow \quad \nearrow \\ \quad \mathcal{B} \\ \quad \nearrow \quad \searrow \\ \text{---} \mathcal{A} \text{---} \end{array} \quad \text{and} \quad \varepsilon = \begin{array}{c} \text{---} \mathcal{B} \text{---} \\ \quad \nearrow \quad \searrow \\ \quad \mathcal{A} \\ \quad \nearrow \quad \searrow \\ \text{---} F \quad G \text{---} \end{array} \quad (6.10)$$

The equalities (6.9) amount to the identifications

$$\begin{array}{ccc} \begin{array}{c} \text{---} F \text{---} \\ \quad \nearrow \quad \searrow \\ \quad \nearrow \quad \searrow \\ \text{---} F \text{---} \end{array} & = & \begin{array}{c} \text{---} F \text{---} \\ \quad \nearrow \\ \quad \searrow \\ \text{---} F \text{---} \end{array} \\ \begin{array}{c} \text{---} G \text{---} \\ \quad \searrow \quad \nearrow \\ \quad \searrow \quad \nearrow \\ \text{---} G \text{---} \end{array} & = & \begin{array}{c} \text{---} G \text{---} \\ \quad \searrow \\ \quad \nearrow \\ \text{---} G \text{---} \end{array} \end{array} \quad (6.11)$$

of diagrams.

In general, the existence of a left adjoint functor does not imply the existence of a right adjoint functor. Even if both adjoints exist, they need not coincide. The same statements hold for left and right adjoints of a 1-morphism in an arbitrary bicategory. It therefore makes sense to give the

**Definition 6.1.** A 1-morphism  $F$  in a bicategory is called *biadjoint* to a 1-morphism  $G$  iff it is both a left and a right adjoint of  $G$ . Since then  $G$  is both left and right adjoint to  $F$  as well, such a pair  $(F, G)$  of 1-morphisms is called a *biadjoint pair*. The adjunction  $(F, G)$  is then called *ambidextrous*.

For a biadjoint pair  $(F, G)$ , we thus have, apart from the 2-morphisms  $\eta$  and  $\varepsilon$  introduced in formula (6.9) additional 2-morphisms

$$\tilde{\eta}: Id_{\mathcal{B}} \Rightarrow FG \quad \text{and} \quad \tilde{\varepsilon}: GF \Rightarrow Id_{\mathcal{A}} \quad (6.12)$$



satisfying zigzag identities analogous to the identities (6.11) for  $\eta$  and  $\varepsilon$ .

Once we restrict ourselves to string diagrams in which all lines are labeled by 1-morphisms admitting an ambidextrous adjoint, and having fixed adjunction 2-morphisms, we can allow for lines with U-turns in string diagrams with the appropriate one of the four adjunction 2-morphisms at the cups and caps, because the relations we have just presented allow us to consistently apply isotopies to all lines. We thus obtain complete isotopy invariance.

For a biadjoint pair  $(F, G)$  one can in particular form the composition  $\tilde{\varepsilon} \circ \eta$ , which is an endomorphism of the identity functor  $Id_{\mathcal{A}}$ , as well as  $\varepsilon \circ \tilde{\eta}$  which is an endomorphism of  $Id_{\mathcal{B}}$ . Graphically,

$$\begin{array}{ccc} \text{-----} & & \text{-----} \\ \tilde{\varepsilon} \circ \eta = & \begin{array}{c} \mathcal{A} \\ \curvearrowright \\ \mathcal{B} \\ \curvearrowleft \end{array} & \varepsilon \circ \tilde{\eta} = \begin{array}{c} \mathcal{B} \\ \curvearrowright \\ \mathcal{A} \\ \curvearrowleft \end{array} \\ \text{-----} & & \text{-----} \end{array} \quad (6.13)$$

It should be appreciated that if the adjunction 2-morphisms are rescaled, the endomorphisms of  $Id_{\mathcal{A}}$  and  $Id_{\mathcal{B}}$  which appear here get rescaled by reciprocal factors.

## 6.2 Frobenius algebras from string diagrams

So far our discussion concerned general bicategories. We now turn to the bicategory of surface operators separating modular tensor categories. As was argued in [KS2], from a surface defect  $S$  separating a modular tensor category  $\mathcal{C}$  from itself, we expect to be able to construct a symmetric special Frobenius algebra in  $\mathcal{C}$  for each Wilson line separating  $S$  and the transparent defect  $T_{\mathcal{C}}$ . Different Wilson lines should yield Morita equivalent Frobenius algebras. We now give a proof of this fact that is based on our description of surface defects.

We thus consider a  $\mathcal{C}$ -module  $S$ . Recall that a Wilson line  $M \in \mathcal{H}om(S, T_{\mathcal{C}})$  is described by a  $\mathcal{C}$ -module functor  $M: S \rightarrow \mathcal{C}$ .

**Lemma 6.2.** *Let  $\mathcal{C}$  be a modular tensor category,  $S$  an object in  $\mathcal{C}\text{-Mod}$  and  $M \in \mathcal{H}om_{\mathcal{C}}(S, T_{\mathcal{C}})$ . Then the functor  $M$  has a biadjoint.*

*Proof.*  $M$  is an additive functor between semisimple  $\mathbb{C}$ -linear categories. Now as an abelian category, a finitely semisimple  $\mathbb{C}$ -linear category is equivalent to  $(\mathcal{V}ect_{\mathbb{C}})^{\boxtimes n}$  with  $n = |\text{Irr}(\mathcal{C})|$  the (finite) number of isomorphism classes of simple objects. Moreover, any additive endofunctor  $F$  of  $\mathcal{V}ect_{\mathbb{C}}$  is given by tensoring with the vector space  $V = F(\mathbb{C})$ , and it is ambidextrous, the adjoint being given by tensoring with the dual vector space  $V^*$ . It follows that the functor  $M$  is equivalent to a functor  $\tilde{M}: (\mathcal{V}ect_{\mathbb{C}})^{\boxtimes n} \rightarrow (\mathcal{V}ect_{\mathbb{C}})^{\boxtimes m}$  for some integers  $n$  and  $m$  and is completely specified by an  $n \times m$ -matrix of  $\mathbb{C}$ -vector spaces. Further, both the left and the right adjoint functor to  $\tilde{M}$  are then given by the ‘adjoint’ matrix, and hence  $\tilde{M}$  is ambidextrous. As a consequence,  $M$  is ambidextrous as well.  $\square$

Since all adjunctions involved are ambidextrous, we can from now on freely use isotopies in the manipulations of string diagrams. We next consider the following construction for any module functor  $M \in \mathcal{H}om_{\mathcal{C}}(S, T_{\mathcal{C}})$ . Denote by  $\bar{M}$  the module functor biadjoint to  $M$  and set  $A^M := M \circ \bar{M}$ . Then  $A^M \in \mathcal{H}om(T_{\mathcal{C}}, T_{\mathcal{C}})$ , which by Proposition 2.4 is equivalent to  $\mathcal{C}$  as

a monoidal category. We proceed to equip the object  $A^M \in \mathcal{C}$  with the structure of a Frobenius algebra in  $\mathcal{C}$ . For the product, we introduce the morphism  $m_{A^M} : A^M \otimes A^M \rightarrow A^M$  as

$$m_{A^M} := \begin{array}{c} \begin{array}{ccccc} & M & & \bar{M} & \\ & \text{---} & & \text{---} & \\ T_{\mathcal{C}} & \downarrow & & \downarrow & T_{\mathcal{C}} \\ & S & & & \\ & \uparrow & & \uparrow & \\ & T_{\mathcal{C}} & & T_{\mathcal{C}} & \\ & \text{---} & & \text{---} & \\ M & \bar{M} & M & \bar{M} & \end{array} \end{array} \quad (6.14)$$

in terms of string diagrams. Similarly, we introduce the coproduct  $\Delta_{A^M} : A^M \rightarrow A^M \otimes A^M$  as

$$\Delta_{A^M} := \begin{array}{c} \begin{array}{ccccc} M & \bar{M} & M & \bar{M} & \\ \text{---} & & \text{---} & & \\ T_{\mathcal{C}} & \downarrow & & \downarrow & T_{\mathcal{C}} \\ & S & & & \\ & \uparrow & & \uparrow & \\ & T_{\mathcal{C}} & & T_{\mathcal{C}} & \\ & \text{---} & & \text{---} & \\ M & \bar{M} & & & \end{array} \end{array} \quad (6.15)$$

The morphisms for counit  $\varepsilon_{A^M}$  and unit  $\eta_{A^M}$  are given by

$$\begin{array}{ccc} \begin{array}{c} \text{---} \\ T_{\mathcal{C}} \\ \text{---} \\ M \quad \bar{M} \end{array} & \begin{array}{c} M \quad \bar{M} \\ \text{---} \\ S \\ \text{---} \\ T_{\mathcal{C}} \end{array} \\ \varepsilon_{A^M} := & \eta_{A^M} := & \end{array} \quad (6.16)$$

It should be appreciated that rescaling the adjunction morphisms rescales product and coproduct, and unit and counit, by inverse factors.

**Proposition 6.3.** *Let  $S$  be an object in  $\mathcal{C}\text{-Mod}$  corresponding to an exact module category. Then for any  $M \in \mathcal{H}\text{om}(S, \mathcal{C})$  the morphisms  $m_{A^M}$ ,  $\eta_{A^M}$ ,  $\Delta_{A^M}$  and  $\varepsilon_{A^M}$  just introduced endow the object  $A^M$  with the structure of a symmetric Frobenius algebra in  $\mathcal{C}$ .*

*Proof.* The equality

$$\begin{array}{ccc} \begin{array}{ccccc} M & & \bar{M} & & \\ \text{---} & & \text{---} & & \\ T_{\mathcal{C}} & \downarrow & & \downarrow & T_{\mathcal{C}} \\ & S & & & \\ & \uparrow & & \uparrow & \\ & T_{\mathcal{C}} & & T_{\mathcal{C}} & \\ & \text{---} & & \text{---} & \\ M & \bar{M} & M & \bar{M} & \end{array} & = & \begin{array}{ccccc} M & & \bar{M} & & \\ \text{---} & & \text{---} & & \\ T_{\mathcal{C}} & \downarrow & & \downarrow & T_{\mathcal{C}} \\ & S & & & \\ & \uparrow & & \uparrow & \\ & T_{\mathcal{C}} & & T_{\mathcal{C}} & \\ & \text{---} & & \text{---} & \\ M & \bar{M} & M & \bar{M} & \end{array} \end{array} \quad (6.17)$$

which follows from the properties of string diagrams, shows that the product is associative. Coassociativity of  $\Delta_{A^M}$  is seen in an analogous manner. The equalities

prove the Frobenius property.

Finally,  $\mathcal{C}$  is rigid, and left and right duals coincide. It is not difficult to see that by construction the algebra  $A^M$  is equal to its dual. The compositions

$$\varepsilon_{A^M} \circ m_{A^M} : A^M \otimes A^M \rightarrow \mathbf{1} \quad \text{and} \quad \Delta_{A^M} \circ \eta_{A^M} : \mathbf{1} \rightarrow A^M \otimes A^M \quad (6.19)$$

give the duality morphisms.  $\square$

There exists a particularly interesting subclass of surface defects for which the Frobenius algebra  $A^M$  obtained from any Wilson line has additional properties. We need first the

**Definition 6.4.** A surface defect  $S$  in  $\mathcal{C}\text{-Mod}$  is said to be *special* iff

$$2\text{-Hom}(Id_S, Id_S) \simeq \mathbb{C}. \quad (6.20)$$

The transparent Wilson line inside a special surface defect  $S$  can only have multiples of the identity as insertions. Put differently, there are no non-trivial local excitations on a surface defect of type  $S$  other than those related to Wilson lines and their junctions.

As an application of this definition, we consider the following situation in a special surface defect: there is a hole punched out, i.e. the surface contains a disk labeled by the transparent defect  $T_C$ ; the label for the boundary of the disk is a Wilson line  $M$ . Since there are no local excitations, we can replace the punched-out hole by the surface defect  $S$ , provided that we multiply every expression obtained with this replacement by a scalar factor depending on the Wilson line  $M$ . For the moment we cannot yet tell whether this scalar factor is non-zero.

Wilson lines separating special defects from the transparent defect should yield Frobenius algebras with a particular property. Recall from section 2 that a *special algebra*  $A$  in a monoidal category  $\mathcal{C}$  is an object which is both an algebra and a coalgebra and satisfies

$$\varepsilon \circ \eta = \beta_1 id_{\mathbf{1}} \quad \text{and} \quad m \circ \Delta = \beta_A id_A \quad (6.21)$$

with non-zero complex numbers  $\beta_1$  and  $\beta_A$ .

**Proposition 6.5.** *Let  $S$  be a special surface defect described by a semisimple module category  $S$  over a modular tensor category  $\mathcal{C}$ . Then for any Wilson line  $M \in \text{Hom}(S, \mathcal{C})$  the corresponding symmetric Frobenius algebra  $A^M$  in  $\mathcal{C}$  is a special algebra.*

*Proof.* For the algebra  $A^M$ , the composition  $m \circ \Delta$  of product and coproduct is described by a string diagram with a hole in the surface defect  $S$  whose boundary is labeled by  $M$ . Since  $S$  is special, there are no local excitations and thus the diagram can be replaced, up to a scalar factor  $\beta_{A^M}$ , by a diagram without hole. On the other hand, since the tensor unit of the modular tensor category  $\mathcal{C}$  is simple, the composition  $\varepsilon \circ \eta$  of counit and unit of  $A^M$  is a multiple  $\beta_1$  of the identity morphism  $\text{id}_1$ .

Both  $\beta_1$  and  $\beta_{A^M}$  depend on the choices of adjunction and coadjunction 2-morphisms for  $\bar{M}$ . However, computing the quantum dimension of  $A^M$  using the duality morphisms (6.19), we obtain

$$\dim(A^M) = \beta_1 \beta_{A^M}, \quad (6.22)$$

so that the product of the two scalars is independent of the choices of adjunction 2-morphisms. Since the object  $A^M$  has been constructed as a composition of a non-vanishing module functor and its adjoint,  $A^M$  is not the zero object. As the only object of a modular tensor category having vanishing quantum dimension is the zero object, we conclude that both scalars  $\beta_1$  and  $\beta_{A^M}$  are non-zero. Hence the algebra  $A^M$  is special.  $\square$

We next investigate how the Frobenius algebras  $A^M$  for a fixed surface defect  $S$  depends on the choice of Wilson line  $M$ .

**Proposition 6.6.** *Let  $S$  be a surface defect in  $\mathcal{C}\text{-Mod}$  and let  $M, M' \in \mathcal{H}\text{om}_{\mathcal{C}}(S, \mathcal{C}_c)$  be Wilson lines separating  $S$  from the transparent defect  $T_c$ . Then the symmetric Frobenius algebras  $A^M$  and  $A^{M'}$  are Morita equivalent.*

*Proof.* We explicitly construct a Morita context. Consider the objects

$$B := M \circ \bar{M}' \quad \text{and} \quad \tilde{B} := M' \circ \bar{M} \quad (6.23)$$

in  $\mathcal{E}nd_{\mathcal{C}}(\mathcal{C}_c) \simeq \mathcal{C}$ . The counit of the adjunction for  $M$  provides a morphism

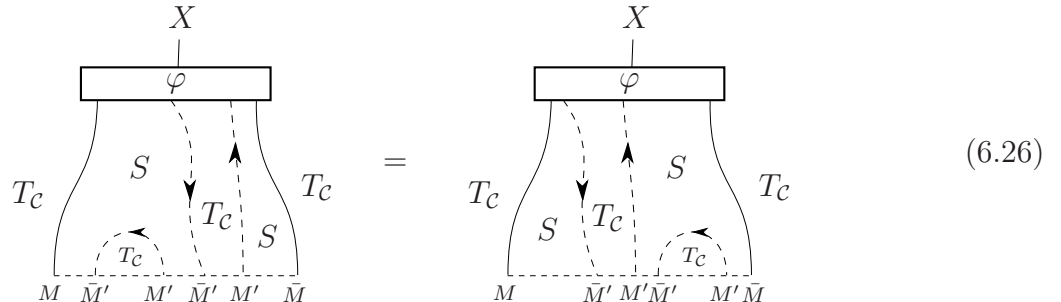
$$M \circ \bar{M} \circ M \circ \bar{M}' \rightarrow M \circ \bar{M}', \quad \text{i.e.} \quad A^M \otimes B \rightarrow B. \quad (6.24)$$

With the help of the isotopy invariance of string diagrams, one quickly checks that this morphism obeys the axiom for a left action of  $A^M$  on  $B$ . This type of argument can be repeated to show that  $B$  has a natural structure of an  $A^M$ - $A^{M'}$ -bimodule, and that  $\tilde{B}$  has the structure of an  $A^{M'}$ - $A^M$ -bimodule.

We next must procure an isomorphism  $B \otimes_{A^{M'}} \tilde{B} \rightarrow A^M$  of bimodules. This is achieved by showing that the morphism

$$M \circ \bar{M}' \circ M' \circ \bar{M} \rightarrow M \circ \bar{M}, \quad \text{i.e.} \quad B \otimes \tilde{B} \rightarrow A^M \quad (6.25)$$

that is provided by the counit of the adjunction (which is obviously a morphism of bimodules) has the universal property of a cokernel. To this end we select any morphism  $\varphi: B \otimes A^{M'} \otimes \tilde{B} \rightarrow X$ , with  $X$  any object of  $\mathcal{C}$ , such that



$$(6.26)$$

We are looking for a morphism  $\tilde{\varphi}: B \rightarrow X$  such that  $\tilde{\varphi} \circ (id_M \otimes \varepsilon_{M'} \otimes id_{\bar{M}}) = \varphi$ . Composing this equality with the morphism  $id_M \otimes \eta_{M'} \otimes id_{\bar{M}}$  yields

$$(6.27)$$

The left hand side of this equality equals  $\beta_{A^{M'}} \tilde{\varphi}$ . This shows that the morphism  $\tilde{\varphi}$  is uniquely determined. To establish that  $A^M$  is indeed a cokernel, we have to show that the morphism  $\beta_{A^{M'}}^{-1} \varphi \circ [id_M \otimes (\eta_{M'} \circ \varepsilon_{M'}) \otimes id_{\bar{M}}]$ , which is the composition of  $\tilde{\varphi}$  with the cokernel morphism, equals  $\varphi$ . This is established by

$$(6.28)$$

Here in the first equality a right action of  $A^{M'}$  composed with  $\varphi$  is replaced by a left action, as in (6.26). The second equality uses the fact that the defect  $S$  is special, so as to remove the bubble at the expense of a factor of  $\beta_{A^{M'}}$ . A similar argument shows that  $\tilde{B} \otimes_{A^M} B \cong A^{M'}$ . This completes the proof.  $\square$

We can summarize the findings of this section in the

**Theorem 6.7.** *Consider the three-dimensional topological field theory corresponding to a modular tensor category. To any surface defect separating the TFT from itself there is associated a Morita equivalence class of special symmetric Frobenius algebras.*

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